CLASS NOTES PART II (M427K)

We just got through talking about n^{th} order linear differential equations, but only when the RHS=0. What do we do when the RHS is NOT zero? Before I charge off into this area, we need to finish up some formal material about linear independence. This really belongs back where we were talking about RHS=0, but it's sort of a subject all its own.

1. Linear Independence

In class, Dr. Guy took a very formal perspective on linear independence. It's really a large topic, and there is no way to do it justice. Basically, the point is that, when we solved (in the last set of notes) n^{th} order linear differential equations with RHS=0, we needed to have *n linearly independent solutions,* and then we just put arbitrary coefficients out front and added them together.

For example, in the 2^{nd} order equation

$$y'' + 5y' + 6y = 0,$$

the solution was just $y = C_1 e^{-2x} + C_2 e^{-3x}$. We needed, however, to verify that e^{-2x} and e^{-3x} were *linearly independent*. Before, we just sort of looked at them and said that they look different.

Now we need to be more precise. If I have a set of functions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ (in our example the set is just two functions, namely $\{e^{-2x}, e^{-3x}\}$) then they are said to be *linearly independent* if I *CANNOT* write any one of them in terms of the other ones. This is just another way of saying that they're all different from one another.

More formally, if I consider the equation

$$C_1 f_1(x) + C_2 f_2(x) + \cdots + C_n f_n(x) = 0$$

then $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ are *linearly independent* if the only way to make this a TRUE equation is to make all of the coefficients $C_1 = C_2 = \ldots = 0$. Otherwise, if one of them is NONZERO (say C_1) then I could write $f_1(x)$ in terms of the other ones:

$$f_1(x) = \frac{1}{C_1}(-C_2 f_2(x) - \cdots - C_n f_n(x)).$$

See what I mean? This is not linearly independent, so it's called *linearly DEPENDENT*. Somehow $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ doesn't give us any more information than $\{f_2(x), f_3(x) \ldots, f_n(x)\}$ (I just removed $f_1(x)$ from the second set), because it can be written in terms of the other ones.

We don't want that!!! We want, for an n^{th} order linear differential equations with RHS=0, exactly n "different" (linearly independent) solutions. So if we find n solutions, then we need some tests to tell if they really are different (otherwise we need to keep looking for more solutions).

Usually you will have to use some kind of algebraic facts about polynomials, sines, cosines, and exponents (like an exponent cannot be written as a polynomial, etc) to prove linear independence.

One nice test is the Wronskian test. This just says that, if the Wronskian is not zero (at least somewhere) then the set is linearly independent. (Unfortunately, if the Wronskian is zero EVERYWHERE, then this test says absolutely nothing). I'm not going to derive this for you. You must believe it.

For examples on this stuff, see Homeworks #13 and #14

link `www.math.utexas.edu/~stirling/teaching/M427K/hw13sols.pdf`

link `www.math.utexas.edu/~stirling/teaching/M427K/hw14sols.pdf`

2. N^{th} Order Linear Differential Equations (constant coeffs) RHS$\neq 0$

We already learned how to solve these types of equations when the RHS=0, but what happens when the RHS$\neq$ 0? You probably already know, so I'm not going to labor over the details, but I'm just going to give you a reminder. Consider the differential equation

$$y''' + 7y' + 3y = e^{7x} + \cos(4x).$$

We solve this in two parts. First, we set the RHS=0 and solve for y_c (using the techniques that we just talked about). So this means that we solve

$$y''' + 7y' + 3y = 0.$$

In this case, this is a 3^{rd} order linear differential equation with RHS=0, so the solution will be the linear combination (added together with arbitrary constants) of 3 linearly independent solutions, like maybe

$$y_c(x) = C_1 e^{-5x} + C_2 x e^{-5x} + C_3 e^{-7x}$$

(please note that this is not the actual solution - I didn't bother solving the differential equation, I'm just writing out the forms so that you can remember the idea).

Technically, we need to check (using what I talked about in the last section) if $\{e^{-5x}, xe^{-5x}, e^{-7x}\}$ are linearly independent. See that section for details. I might try the Wronskian method for this (because I *think* that they *are* linearly independent, but I need to prove it).

Second, we try to find ANY "old dinky *particular* solution" $y_p(x)$ to the original equation $y''' + 7y' + 3y = e^{7x} + \cos(4x)$. This can be done using 3 methods: clever guessing, Heaviside method, or Variational parameters. I'll mention all 3 here briefly.

Finally, once we have $y_c(x)$ and $y_p(x)$, then the full solution is just $y(x) = y_c(x) + y_p(x)$. I'll bet that you remember this, but it's good to be reminded over and over.

So let's briefly mention the first method for solving for $y_p(x)$: clever guessing

2.1. Clever Guessing.

All we're doing here is guessing what the solution will "look like", and then plugging it in and adjusting the constants so that it is right. This won't always work. I don't want to labor over the details here, but here's an example. Consider the diffeq

$$y'' + 7y = e^{7x} + \cos(4x) + 1 + x^2.$$

I would *guess* that a particular solution will look like

$$y_p(x) = Ae^{7x} + B\sin(4x) + C\cos(4x) + D + Ex + Fx^2.$$

Then I would plug this guess into the diffeq and adjust the constants A, B, C, D, E, F so that the LHS looks like the RHS exactly. I hope that you recognize this pattern. I'm being very quick.

2.2. Heaviside Method. Let's look at the diffeq again:

$$y'' + 7y = e^{7x} + \cos(4x) + 1 + x^2.$$

I can just rewrite this in "big D" notation like

$$(D^2 + 7)y(x) = e^{7x} + \cos(4x) + 1 + x^2.$$

So then if I want to solve for $y_p(x)$, I just "divide" by $(D^2 + 7)$,giving

$$y_p(x) = \frac{1}{D^2+7}\left(e^{7x} + \cos(4x) + 1 + x^2\right).$$

So now I just need to tell you how $\frac{1}{P_n(D)}$ acts on functions like e^{7x} and $\cos(4x)$ (where $P_n(D)$ is a polynomial in the symbol D). We can't do ALL functions, but we can do many important ones. Here are the rules

$0'$	$\frac{1}{D}f(x) = \int f(x)dx$
$1'$ $P_n(a) \neq 0$	$\frac{1}{P_n(D)}e^{ax} = e^{ax}\frac{1}{P_n(a)}$
$2'$ $P_n(-a^2) \neq 0$	$\frac{1}{P_n(D^2)}\sin(ax) = \sin(ax)\frac{1}{P_n(-a^2)}$
$3'$ $P_n(-a^2) \neq 0$	$\frac{1}{P_n(D^2)}\cos(ax) = \cos(ax)\frac{1}{P_n(-a^2)}$
$4'$	$\frac{1}{P_n(D)}(e^{ax}V(x)) = e^{ax}\frac{1}{P_n(D+a)}V(x)$
$5'$ $\phi(a) \neq 0$	$\frac{1}{\phi(D)(D-a)^n}e^{ax} = e^{ax}\frac{x^n}{\phi(a)n!}$
$6'$	$\frac{1}{D^2+a^2}\sin(ax) = -\frac{x}{2a}\cos(ax)$
$7'$	$\frac{1}{D^2+a^2}\cos(ax) = \frac{x}{2a}\sin(ax)$
$8'$ Red Rover	see your notes for this trick

That's it!!! This should be enough tools. For examples see Homeworks #9, #10, #11, and #12.

2.3. Variational Parameters. This is the last method we will cover for solving for $y_p(x)$. I'm too busy to write it up right now, so just consult Homework #15

3. N^{th} Order Linear DiffEq's WITHOUT CONSTANT COEFFICIENTS

Now we've spent a lot of time on linear differential equations, but they always had constant coefficients, e.g.

$$y''(x) + 5y'(x) + 7y(x) = e^{7x}.$$

Now the question is, what do I do if the "5" or the "7" in the above equation are NOT just lousy old constants, but instead are functions (of x)? Maybe I have

$$x^3y''(x) + (x + x^2)y'(x) + \cos(x)y(x) = e^{7x}.$$

Now how do I solve these?

3.1. Power Series Method. See Homework #16
link `www.math.utexas.edu/~stirling/teaching/M427K/hw16.pdf`