HOMEWORK #9 (M427K FALL 2004)

Introduction

You are supposed to find the "particular" solution y_p to the following differential equations (NOTE that the full solution to these differential equations would be $y_p + y_c$ where y_c is the solution to the differential equation if the RHS=0!!!).

1. Find the Particular Solution to the Differential Equation

$$(D-3)(D+4)(D-6)y(x) = 4e^{5x} + 6e^{7x} + 8e^{-9x}$$

2. Find the Particular Solution to the Differential Equation

$$D(D-1)(D-2)(D+6)y(x) = e^{4x} + e^{-4x} + 6e^{\pi x} + 7e^{\sqrt{2}x} + \sum_{n=1}^{1000} e^{7nx} \tag{2.1}$$

2.1. Solution. This is solved using the so-called "Heaviside Method". You may try to look this method up under the name "Laplace Transform" if you wish. Essentially, we are going to treat the differential equation like it is just an algebra equation.

So first consider the *operator* $D(D-1)(D-2)(D+6)$ which "operates" on $y(x)$. This is kind of a different way of thinking about derivatives, as something that "eats" a function $y(x)$ and "poops out" another function (yeah, it's late). If we are very naive, we think that we should be able to define some kind of "inverse" or "undo" operator which does the opposite - it takes the poop and turns it back into $y(x)$. So it basically "undoes" what $D(D-1)(D-2)(D+6)$ does. So the notation that we would choose for this operator is

$$\frac{1}{D(D-1)(D-2)(D+6)}$$

(since if we multiply it by $D(D-1)(D-2)(D+6)$ then we just get "1", the identity operator (the operator that does nothing)).

So if we multiply both sides of equation 2.1 by this "inverse operator" (this is the technical name for the "undo" operator) then we get the equation

$$y_p(x) = \frac{1}{D(D-1)(D-2)(D+6)} \left\{ e^{4x} + e^{-4x} + 6e^{\pi x} + 7e^{\sqrt{2}x} + \sum_{n=1}^{1000} e^{7nx} \right\}. \tag{2.2}$$

That's IT!!! We have a *particular* solution (REMEMBER we still need to solve for the GENERAL solution by adding this particular solution y_p to the solution y_c when the RHS=0, but I won't do it here!!!)

Wait a minute... I haven't told you anything. I haven't even told you if this inverse operator even EXISTS (or what it is, for that matter)!!!

The intuitive jump made by Heaviside is that it DOES exist. We will only know, however, how it operates on certain functions (exponentials, sines, and cosines). So we *won't know everything* about it, but we will know enough to solve problems if the RHS is made up of exponentials, sines, and cosines.

So we just need to figure out what $\frac{1}{D(D-1)(D-2)(D+6)}$ does to the exponentials in equation 2.2. Everything done here is in quotation marks.

First, we know how to take derivatives if the D's are only in the numerator (it's just a regular derivative): $D(D-1)(D-2)(D+6)e^{ax} = e^{ax}a(a-1)(a-2)(a+6)$.

Now, if $a(a-1)(a-2)(a+6)$ is not zero, then we can divide both sides by it and we can multiply both sides by the "undo" operator $\frac{1}{D(D-1)(D-2)(D+6)}$ to yield the following equation

$$\frac{1}{D(D-1)(D-2)(D+6)}e^{ax} = e^{ax}\frac{1}{a(a-1)(a-2)(a+6)}.$$

That's REALLY it! Everything done above was not quite right (it's not rigorous, as mathematicians would say), but it's a good enough pneumonic device to remember the argument.

So applying this result to equation 2.2 gives

$$\begin{aligned} y_p(x) &= \tfrac{e^{4x}}{4(4-1)(4-2)(4+6)} + \tfrac{e^{4x}}{-4(-4-1)(-4-2)(-4+6)} + \tfrac{6e^{\pi x}}{\pi(\pi-1)(\pi-2)(\pi+6)} \\ &+ \tfrac{7e^{\sqrt{2}x}}{\sqrt{2}(\sqrt{2}-1)(\sqrt{2}-2)(\sqrt{2}+6)} + \sum_{n=1}^{1000} \tfrac{e^{7nx}}{7n(7n-1)(7n-2)(7n+6)} \end{aligned}.$$

I assume that you can simplify this yourself. Again, remember that this is only the particular solution. The full solution would be $y_p + y_c$ where y_c is the solution to equation 2.1 (EXCEPT I MAKE THE RHS=0!!!)