HOMEWORK #3 SOLUTIONS (M427K FALL 2004)

1. Solve Differential Equation

$$(x + y + 3)dx + (2x - 3y + 2)dy = 0$$

(hint: "kissing cousin" of the homogeneous equation. try substituting $x = \overline{x} + h$ and $y = \overline{y} + k$ to make it into a homogeneous equation. Then solve homogeneous equation as in Homework #2)

1.1. Solution.

This equation looks *almost* homogeneous, but not quite (remember the crossing lines). Try substituting $x = \overline{x} + h$ and $y = \overline{y} + k$ where h and k are constants to be determined. We have to reformulate the differentials... fortunately this is easy. It is just $dx = d(\overline{x} + h) = d\overline{x}$ (and $dy = d\overline{y}$ similarly). So plug this junk into the original equation to give (reordering a little)

$$(\overline{x} + \overline{y} + h + k + 3)d\overline{x} + (2\overline{x} - 3\overline{y} + 2h - 3k + 2)d\overline{y} = 0.$$

In order to make this a homogeneous equation (as in homework set #2) we need to get rid of the pieces that are hanging out (set the unwanted junk equal to zero). So we must solve simultaneously the equations

$$\begin{array}{rcl} h + k + 3 & = & 0 \\ 2h - 3k + 2 & = & 0 \end{array}.$$

Note that this is not always possible (if lines are parallel), but in this case it is, giving $h = -11/5$ and $k = -4/5$.

So choosing these values for h and k leaves me with the equation

$$(\overline{x} + \overline{y})d\overline{x} + (2\overline{x} - 3\overline{y})d\overline{y} = 0. \tag{1.1}$$

Now I'm sick of typing the "little bars" over all of the x's and y's, so I'll *leave them out for now* (they should be in there, of course). I'll replace them at the end.

Now this is precisely a homogeneous equation, so we can solve it by introducing a new variable u and substituting $y = ux$ (which means that $dy = udx + xdu$). Putting this into equation 1.1 gives (multiplying everything out and grouping all of the dx terms together and du terms together):

$$(1 + 3u - 3u^2)dx + (2 - 3u)xdu = 0$$

(Note that I avoided a lot of annoying algebra by dividing both sides of the equation by x before I made the substitution).

This equation is now variable separable. Divide the entire equation by $x(1 + 3u - 3u^2)$ to separate, giving the final expression

$$\frac{dx}{x} + \frac{2 - 3u}{1 + 3u - 3u^2}du = 0.$$

Now you should be able to integrate.

Integrating the first term just gives $\ln | x |$. The second term is kind of a bitch. Here's how I'm doing it (there are probably 5 good ways to do it... I'd invite you to experiment, but I know that you won't).

I notice that in the integral

$$\int \frac{2-3u}{1+3u-3u^2}du$$

the numerator looks *almost* like the derivative of the denominator. So let me play with the constants (without changing the overall expression, mind you) so that it DOES (and then we're going to get another correction term to counteract our playing). Try

$$\frac{1}{2}\int \frac{(3-6u)+1}{1+3u-3u^2}du$$

Go ahead and multiply it out if you don't believe me that it's the same! Well, I know that you won't, so let's proceed. This expression is just

$$\frac{1}{2}\left\{\int \frac{3-6u}{1+3u-3u^2}du + \int \frac{1}{1+3u-3u^2}du\right\}. \tag{1.2}$$

Now the first term in this expression looks like

$$\int \frac{df}{f}$$

where I define $f = 1+3u-3u^2$ (so $df = (3-6u)du$). So that's easy to integrate... it's just $\ln \mid f \mid$(which $= \ln \mid 1+3u-3u^2 \mid$).

The second term in equation 1.2 can be integrated by completing the square in the denominator. If you don't know how to do this, then you're out of luck. I'm not going to hold your hand. You can just check that these two are the same by multiplying the stuff in the RHS out

$$\int \frac{1}{1+3u-3u^2}du = -\frac{1}{3}\int \frac{1}{(u-\frac{1}{2})^2 - \left(\sqrt{\frac{7}{12}}\right)^2}du.$$

So this is in the *form*

$$\int \frac{dg}{g^2-a^2}$$

where $g = u - \frac{1}{2}$ and $a = \sqrt{\frac{7}{12}}$. The solution of THIS integral is (look it up)

$$\frac{1}{2a}\{\ln \mid g-a \mid - \ln \mid g+a \mid\}.$$

So, wrapping up, the second term in equation 1.2 is given by

$$\frac{1}{2\sqrt{\frac{7}{12}}}\left\{\ln \mid u - \frac{1}{2} - \sqrt{\frac{7}{12}} \mid - \ln \mid u - \frac{1}{2} + \sqrt{\frac{7}{12}} \mid\right\}.$$

Collecting all of these results (and now putting the bars back in over the x's) the solution of the differential equation is:

$$\ln \mid \overline{x} \mid + \frac{1}{2}\left\{\ln \mid 1+3u-3u^2 \mid + \left(-\frac{1}{3}\right)\frac{1}{2\sqrt{\frac{7}{12}}}\left\{\ln \mid u - \frac{1}{2} - \sqrt{\frac{7}{12}} \mid - \ln \mid u - \frac{1}{2} + \sqrt{\frac{7}{12}} \mid\right\}\right\} = C.$$

That's it... I expect that you can simplify this junk, get rid of the u's, the $\overline{x}$'s, and the $\overline{y}$'s and rewrite the solution in terms of x and y by yourself (you'd *better* do that).