M3150 PDEs Final (Spencer Stirling) - May 4, 2009

Directions: Use both the front and back of the paper for your solutions.

1) (5 points) Consider the 1-d heat equation on a rod of length $L = 1$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where $u(x,t)$ is an unknown function. Assume that the thermal diffusivity $c^2 = 1$. Let the initial temperature distribution be given by the function

$$u(x,0) = f(x) = \cos(\pi x) \tag{2}$$

Assume that the bar ends are *insulated* (make sure you understand what boundary condition this implies). Using the method of separation of variables (you do not need to repeat the actual separation steps... just make sure that you understand which formulas should be used - this can be tricky) solve the heat equation as a function of time.

The separation is done in 3.6

insulated means $\left.\frac{\partial u}{\partial x}\right|_{x=0} = \left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$

The solution is (after separation)

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t} \cos\left(\frac{n\pi}{L} x\right) \qquad \lambda_n = c\,\frac{n\pi}{L}$$

$$\boxed{u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-(n\pi)^2 t} \cos(n\pi x)}$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \int_0^1 \cos(\pi x)\,dx = \left.\frac{\sin(\pi x)}{\pi}\right|_0^1 = 0$$

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$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx = 2\int_0^1 \underbrace{\cos(\pi x)\cos(n\pi x)}_{\text{even function}}\,dx$$

$$= \int_{-1}^{1} \cos(\pi x)\cos(n\pi x)\,dx = \text{use orthagonality} = \begin{cases} 1 & n=1 \\ 0 & \text{otherwise} \end{cases}$$

so $\boxed{u(x,t) = 1 \cdot e^{-\pi^2 t}\cos(\pi x)}$

2) (2 points) Using the previous problem find the steady-state temperature distribution in the rod (note: it actually isn't necessary to have the solution to the previous problem to figure the answer out - but you MUST JUSTIFY YOUR ANSWER with actual calculations).

there are multiple ways to solve this. $\frac{\partial u}{\partial t} = 0 \leftarrow$ steady state

Take (for heat eqn steady state is $t \to \infty$)

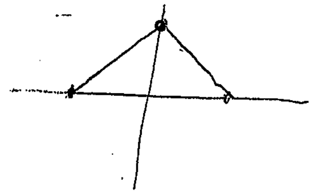
$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} e^{-\pi^2 t} \cos(\pi x) = \boxed{0}$$

Note this is also the average initial temp $\frac{1}{L}\int_0^L f(x)\,dx = \int_0^1 \cos(\pi x)\,dx = 0$

3) (5 points) Find the Fourier integral representation (using sines and cosines, NOT complex exponentials) of the function

$$f(x) = \begin{cases} 1-|x| & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$f(x) = \begin{cases} 0 & x \le -1 \\ 1+x & -1 < x \le 0 \\ 1-x & 0 \le x < 1 \\ 0 & 1 > x \end{cases}$$



$$\boxed{f(x) = \int_0^\infty \left[A(w)\cos wx + B(w)\sin wx\right] dw} \quad (-\infty < x < \infty)$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(wt)\,dt \qquad B(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\sin(wt)\,dt$$

now to compute these

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$$\text{so} \quad A(\omega) = \frac{1}{\pi}\left[\int_{-1}^{0}(1+t)\cos(\omega t)\,dt + \int_{0}^{1}(1-t)\cos(\omega t)\,dt\right]$$

$$= \frac{1}{\pi}\left[\int_{-1}^{1}\cos(\omega t)\,dt + \int_{-1}^{0} t\cos\omega t\,dt - \int_{0}^{1} t\cos\omega t\,dt\right]$$

we need $\int \underset{u}{t}\,\underset{dV}{\cos\omega t}\,dt$ $\quad$ by parts $= \dfrac{t\sin\omega t}{\omega} - \displaystyle\int \frac{\sin\omega t}{\omega}\,dt$

$$= \frac{t\sin\omega t}{\omega} + \frac{\cos\omega t}{\omega^2}$$

$$A(\omega) = \frac{1}{\pi}\left[\frac{\sin\omega t}{\omega}\Big|_{-1}^{1} + \left(\frac{t\sin\omega t}{\omega} + \frac{\cos\omega t}{\omega^2}\right)\Big|_{-1}^{0} - \left(\frac{t\sin\omega t}{\omega} + \frac{\cos\omega t}{\omega^2}\right)\Big|_{0}^{1}\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{2\sin\omega}{\omega}} + \left(\frac{1}{\omega^2} - \left(\cancel{\frac{-1\sin(-\omega)}{\omega}} + \frac{\cos(-\omega)}{\omega^2}\right)\right) - \left(\cancel{\frac{\sin\omega}{\omega}} + \frac{\cos\omega}{\omega^2} - \frac{1}{\omega^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{2}{\omega^2} - \frac{2\cos\omega}{\omega^2}\right] = \boxed{\frac{2}{\pi\omega^2}(1-\cos\omega) = A(\omega)}$$

$$B(\omega) = \frac{1}{\pi}\left[\int_{-1}^{0}(1+t)\sin(\omega t)\,dt + \int_{0}^{1}(1-t)\sin\omega t\,dt\right]$$

$$= \frac{1}{\pi}\left[\int_{-1}^{1}\sin\omega t\,dt + \int_{-1}^{0} t\sin\omega t\,dt - \int_{0}^{1} t\sin\omega t\,dt\right]$$

we need $\int \underset{u}{t}\,\underset{dV}{\sin\omega t}\,dt = -\dfrac{t\cos\omega t}{\omega} + \displaystyle\int \frac{\cos\omega t}{\omega}\,dt = -\frac{t\cos\omega t}{\omega} + \frac{\sin\omega t}{\omega^2}$

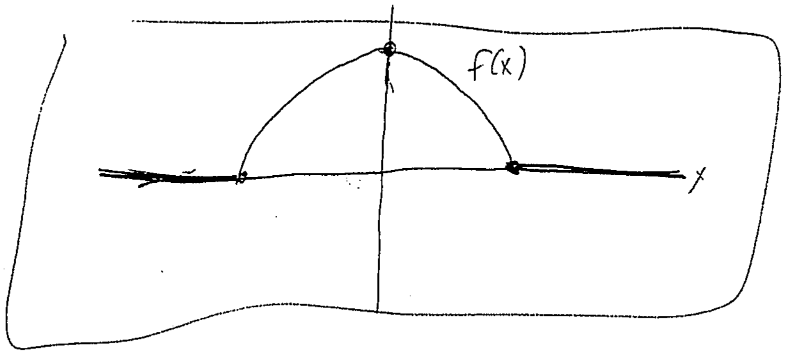
$$\text{so} \quad B(\omega) = \frac{1}{\pi}\left[0 + \left(-\left(\cancel{\frac{\cos(-\omega)}{\omega}} + \frac{\sin(-\omega)}{\omega^2}\right)\right) - \left(\cancel{\frac{-1\cos\omega}{\omega}} + \cancel{\frac{\sin\omega}{\omega^2}}\right)\right)$$

$$\boxed{B(\omega) = 0}$$

(it's an even function)

4) Plot the following function (2 points) and find its Fourier transform (use complex exponential formula) (4 points)

$$f(x) = \begin{cases} 1 - x^2 & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$



$$\mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2) e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-1}^{1} e^{-iwx} dx - \int_{-1}^{1} x^2 e^{-iwx} dx \right]$$

now $\int \underset{u}{x^2} \underset{dV}{e^{-iwx}} dx = \frac{x^2 e^{-iwx}}{-iw} + \int \underset{u}{2x} \underset{dV}{\frac{e^{-iwx}}{iw}} dx$

$$= -\frac{x^2 e^{-iwx}}{iw} + \frac{2}{iw} \left[\frac{x e^{-iwx}}{-iw} + \int \frac{e^{-iwx}}{iw} dx \right]$$

$$= -\frac{x^2 e^{-iwx}}{iw} + \frac{2x}{w^2} e^{-iwx} + \frac{2}{iw^3} e^{-iwx}$$

so $\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\left[\cancel{\frac{1}{-iw}\left(e^{-iw} - e^{iw}\right)} - \left\{\cancel{-\frac{1}{iw}\left(e^{-iw} - e^{iw}\right)}\right.\right.$

$$+\frac{2}{w^2}\left(e^{-iw} + e^{iw}\right) +$$

$$\left.\left.\frac{2}{iw^3}\left(e^{-iw} - e^{iw}\right)\right\}\right]$$

$$\boxed{= \frac{1}{\sqrt{2\pi}}\left[\frac{-4}{w^2}\left(\cos(w) - \frac{\sin(w)}{w}\right)\right] = \hat{f}(w)}$$

5) (5 points) Compute the Fourier transform of the following function (hint: shifting properties of the Fourier transform):

$$f(x) = \frac{\sin(x) + \cos(2x)}{4 + x^2} \tag{5}$$

$$f(x) = \sin x\left(\frac{1}{4+x^2}\right) + \cos(2x)\,\frac{1}{4+x^2}$$

using appendix pg A66 we see $\mathcal{F}\left(\frac{1}{a^2+x^2}\right) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-a|\omega|}}{a}$

$a > 0$

obviously here $a = 2$

Now using 7.2 #20 we have

$$\mathcal{F}(\cos(\alpha x)f(x))(\omega) = \frac{\hat{f}(\omega-\alpha) + \hat{f}(\omega+\alpha)}{2}$$

$$\mathcal{F}(\sin(\beta x)f(x))(\omega) = \frac{\hat{f}(\omega-\beta) - \hat{f}(\omega+\beta)}{2i}$$

putting all together

$$\boxed{\hat{f}(\omega) = \frac{1}{4}\sqrt{\frac{\pi}{2}}\left\{\frac{e^{-2|\omega-1|} - e^{-2|\omega+1|}}{i} + e^{-2|\omega-2|} + e^{-2|\omega+2|}\right\}}$$

6) (5 points) Find the *inverse* Fourier transform of the following function (hint: convolutions might be useful):

$$\hat{h}(\omega) = \begin{cases} \frac{1}{1+\omega^2} & |\omega| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$\hat{h}(\omega) = \underbrace{\left(\frac{1}{1+\omega^2}\right)}_{\hat{f}(\omega)} \hat{g}(\omega) \qquad \text{where } \hat{g}(\omega) = \begin{cases} 0 & \omega \leq -1 \\ 1 & -1 < \omega < 1 \\ 0 & 1 \geq \omega \end{cases}$$

using table of inverse tranforms we see

$$h(x) = f(x) \underset{\text{convolution}}{*} g(x)$$

$$f(x) = \text{inverse transform of } \sqrt{\frac{\pi}{2}}\left(\sqrt{\frac{2}{\pi}}\left(\frac{1}{1+\omega^2}\right)\right)$$

$$= \sqrt{\frac{\pi}{2}}\, e^{-|x|}$$

$$\text{Now } g(x) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{i\omega x}\, d\omega = \frac{1}{\sqrt{2\pi}}\frac{1}{ix}\left[e^{ix} - e^{-ix}\right] = \frac{2}{x\sqrt{2\pi}}\sin(x)$$

$$\text{so } h(x) = \sqrt{\frac{\pi}{2}}\, e^{-|x|} * \frac{2}{x\sqrt{2\pi}}\sin(x) = \boxed{e^{-|x|} * \frac{\sin(x)}{x}}$$

this answer is good enough

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-|s|}\,\frac{\sin(x-s)}{(x-s)}\,ds$$

7) (5 points) Using the Fourier transform method find the solution to the PDE

$$\frac{\partial u}{\partial t} + \sin(t)\frac{\partial u}{\partial x} = 0 \tag{7}$$

with the initial condition

$$u(x,0) = \cos(x) \tag{8}$$

(hint: use FORMAL operations of the Fourier transform - pretend that every function here actually HAS a Fourier transform. Check that your final solution does indeed satisfy the PDE and boundary condition).

$$\tilde{\mathcal{F}}_x\left(\frac{\partial u}{\partial t} + \sin(t)\frac{\partial u}{\partial x}\right) = \mathcal{F}(0)$$

$$\frac{\partial \hat{u}}{\partial t} + \sin(t)\, i\omega\, \hat{u} = 0 \rightarrow \frac{\partial \hat{u}}{\partial t} = -i\omega \sin(t)\, \hat{u}$$

so $\hat{u}(\omega,t) = Ce^{+i\omega\cos(t)}$

now use $u(x,0) = \cos(x) = $ "$f(x)$" to find C

$$\hat{u}(\omega,0) = \hat{f}(\omega) = Ce^{i\omega\cos(0)} = Ce^{i\omega}$$

so $C = \hat{f}(\omega)e^{-i\omega}$

so $\hat{u}(\omega,t) = \hat{f}(\omega)e^{i\omega(\cos(t)-1)}$

so $u(x,t) = \mathcal{F}^{-1}\left(\hat{f}(x)e^{i\omega(\cos(t)-1)}\right)$

$=$ use shifting property $= f(x+(\cos(t)-1))$

so $\boxed{u(x,t) = \cos(x+(\cos(t)-1))}$

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