Solutions

M3150 PDEs midterm 1 (Spencer Stirling) - February 23, 2009

Directions: Use both the front and back of the paper for your solutions. You may attach more sheets if necessary

Consider the partial differential equation

$$e^{x^2}\frac{\partial u}{\partial x} = -x\frac{\partial u}{\partial y} \tag{1}$$

where $u(x, y)$ is an unknown function. There are infinitely many solutions.

1) (1 point) Is this PDE linear or nonlinear? linear

2) (1 point) Is this PDE homogeneous or inhomogeneous? homogeneous

3) (6 points) Find *all* solutions $u(x, y)$ to this PDE using the method of characteristic curves, and in addition write a particular solution down (just pick one).

since e^{x^2} is never zero we can legally divide both sides by it:

$$e^{x^2}\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0 \longrightarrow \frac{\partial u}{\partial x} + e^{-x^2}x\frac{\partial u}{\partial y} = 0$$

$$\vec{\nabla} u \cdot (1, xe^{-x^2}) = 0$$

so $u(x,y)$ is constant along integral curves of vector field $(1, xe^{-x^2})$

$$\frac{dy}{dx} = xe^{-x^2} \longrightarrow \int dy = \int dx\, xe^{-x^2}$$

let $w = x^2 \Longrightarrow dw = 2x\,dx \Longrightarrow x\,dx = \frac{dw}{2}$

place more work here

so $y = \int \frac{dw}{2} e^{-w} = -\frac{e^{-w}}{2} + C = -\frac{e^{-x^2}}{2} + C$

so vector field $(\cancel{1}, xe^{-x^2})$ determines a family of curves in (x,y)-plane given by

$$y = -\frac{e^{-x^2}}{2} + C$$

each C determines a different curve

$$C = y + \frac{e^{-x^2}}{2}$$

so we conclude $u(x,y) = \phi\left(y + \frac{e^{-x^2}}{2}\right)$ where ϕ is an arbitrary differentiable function

Particular sol'n: $\phi(z) = z$
for example

$u(x,y) = y + \frac{e^{-x^2}}{2}$ $\leftarrow$ one solution, for example

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4) (2 points) Determine the fundamental period of the function

$$\exp\left(\frac{1}{2+\sin(\frac{2}{3}x)}\right) \tag{2}$$

SPENCER RULES

see 2.1 exercise 9c

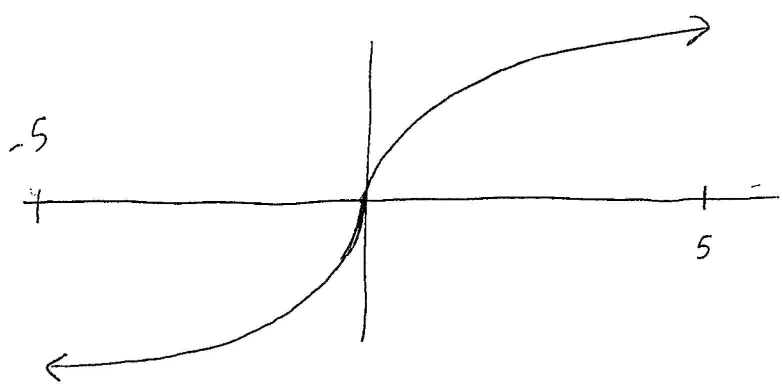
$g(z) = \exp\left(\frac{1}{2+z}\right)$ $g(f(x)) = \exp\left(\frac{1}{2+\sin(\frac{2}{3}x)}\right)$

$f(x) = \sin(\frac{2}{3}x)$

T = period = same as period of $f(x)$: $\frac{2}{3}T = 2\pi \Rightarrow \boxed{T = 3\pi}$

5) (3 points) Determine if the following function is piecewise continuous, piecewise smooth, or neither on the interval $[-5, 5]$. Draw a graph of the function (3 more points).

$$f(x) = x^{\frac{1}{3}} \tag{3}$$



$x^{1/3}$ is inverse function for x^3

It is clearly continuous (it is a rational function w/ no zeros in denominator)

derivative?

$f'(x) = \frac{1}{3x^{2/3}} \leftarrow$ blows up near $x = 0$

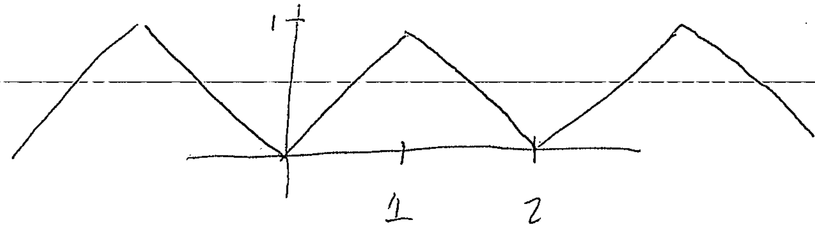
Not smooth not piecewise smooth, either (both left and right limits $\xrightarrow[x\to 0]{}$ blow up)

conclude $\boxed{f(x) = \text{continuous}}$

6) (6 points) Derive the Fourier series and draw the graph (another 2 points) for the function

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2 - x & 1 \le x \le 2 \end{cases} \tag{4}$$

assuming that $f(x)$ has fundamental period $T = 2$. so "$p = T/2 = 1$"



even function so $b_n = 0$, $n \ge 1$

$$a_0 = \frac{1}{T}\int_0^T f(x)\,dx = \frac{1}{2}\left[\int_0^1 x\,dx + \int_1^2 (2-x)\,dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} + \frac{1}{2}\right] = \boxed{\frac{1}{2} = a_0}$$

$$\underset{n \ge 1}{a_n} = \frac{1}{T/2}\int_0^T f(x)\cos\left(n\frac{2\pi}{T}x\right)dx = 1\left[\int_0^1 x\cos(n\pi x)\,dx + \int_1^2 (2-x)\cos(\,)\,dx\right]$$

Look like we'll need this integral a couple of times:

$$\int_a^b \underset{u}{x}\,\underset{dV}{\cos(n\pi x)\,dx} = \left.\frac{x\sin(n\pi x)}{n\pi}\right]_a^b - \int_a^b \frac{\sin(n\pi x)}{n\pi}\,dx = \searrow$$

put more work here

$$\int_a^b x \cos(n\pi x)\,dx = \frac{1}{n\pi}\left[b\sin(n\pi b) - a\sin(n\pi a) + \frac{1}{n\pi}\left(\cos(n\pi b) - \cos(n\pi a)\right)\right]$$

so

$$a_n \underset{n\geq 1}{=} \int_0^1 x\cos(n\pi x)\,dx + \int_1^2 2\cos(n\pi x)\,dx - \int_1^2 x\cos(n\pi x)\,dx$$

$$= \frac{1}{n\pi}\left[\cancel{1\sin(n\pi)}^{0} - \cancel{0\sin(0)}^{0} + \frac{1}{n\pi}\left(\cos(n\pi) - \underbrace{\cos(0)}_{1}\right)\right]$$

$$+ \frac{2}{n\pi}\left(\cancel{\sin(n\pi 2)}^{0} - \cancel{\sin(n\pi)}^{0}\right)$$

$$- \frac{1}{n\pi}\left[2\cancel{\sin(n\pi 2)}^{0} - 1\cancel{\sin(n\pi)}^{0} + \frac{1}{n\pi}\left(\underbrace{\cos(n\pi 2)}_{1} - \cos(n\pi)\right)\right]$$

$$= \frac{2}{(n\pi)^2}\left(\cos(n\pi) - 1\right) = \begin{cases} 0 & n \text{ even} \\ \frac{-4}{(n\pi)^2} & n \text{ odd}\end{cases}$$

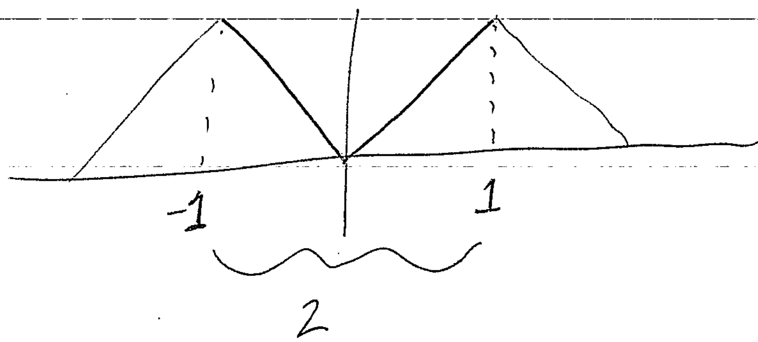
so
$$\boxed{\begin{aligned} f(x) &= \frac{1}{2} + \sum_{\substack{n=1 \\ \text{odd only}}}^{\infty} -\frac{4}{(n\pi)^2}\cos(n\pi x) \\ &= \frac{1}{2} + \sum_{k=0}^{\infty} -\frac{4}{(2k+1)^2\pi^2}\cos((2k+1)\pi x)\end{aligned}}$$

7) (6 points) Find the half-range *even* expansion for the function

$$f(x) = x \qquad 0 \le x \le 1 \tag{5}$$

(hint: most of the integrals were done in problem 6)

I expand to even function with twice period



This is the same function as in # 6!

$$f(x) = \frac{1}{2} + \sum_{k=0}^{\infty} -\frac{4}{(2k+1)^2\pi^2} \cos\left((2k+1)\pi x\right)$$

put more work here

8) (6 points) Find the Fourier series of the following function (use the *complex* exponential form of the Fourier series).

$$f(x) = e^x \qquad -\pi \leq x \leq \pi \tag{6}$$

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}$$

this is a period of length 2π, so no need to alter period in exponential

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{x} e^{-inx}\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{x(1-in)}\,dx$$

$$= \frac{1}{2\pi}\frac{e^{x(1-in)}}{(1-in)}\Big|_0^{2\pi} = \frac{1}{2\pi(1-in)}\left(e^{2\pi}-1\right) = c_n$$

$$\boxed{f(x) = \sum_{n=-\infty}^{+\infty} \frac{e^{2\pi}-1}{2\pi(1-in)} e^{inx}}$$

finished in 40 minutes

put more work here