

WITTEN'S APPROACH TO THE MORSE INEQUALITIES USING ELLIPTIC THEORY

SPENCER STIRLING

ABSTRACT. In this paper we review an approach to Morse theory pioneered by Edward Witten in 1982 [1]. First we start with the statements of classical Morse theory and the Morse inequalities. Next we introduce Witten's idea of a perturbed Laplacian acting on the de Rham complex, and then using the elliptic theory we prove the Morse inequalities.

1. CLASSICAL MORSE THEORY

Let M be a manifold (for simplicity we restrict our attention to compact manifolds). Then a smooth function $h : M \rightarrow \mathbb{R}$ is called a *Morse function* if the critical points m_i are non-degenerate, i.e. if the Hessian¹ at each critical point is a non-degenerate (bilinear symmetric) form.² This implies that a Morse function has *isolated* critical points. Since we restrict our attention to compact manifolds, this also means that we have a finite number of critical points.

The first idea of Morse theory is that the topology of the manifold does not change in between the critical points; i.e., letting $M_c = \{m \in M : h(m) \leq c \in \mathbb{R}\}$,

¹Note that the Hessian (of a function $h : M \rightarrow \mathbb{R}$) on a manifold M *without* a Riemannian metric is defined *only at a critical point* m by

$$H_h(X, Y) = X \cdot (Y \cdot h)(m).$$

This expression is tensorial (i.e. depends only on the values of X and Y at m), which is easily seen by considering

$$H_h(X, fY) = X \cdot (fY \cdot h)(m) = fX \cdot (Y \cdot h)(m) + (X \cdot f)(Y \cdot h)(m).$$

The last term drops out at a critical point, showing that the Hessian is bilinear there. The Hessian is also *symmetric* under exchange of X and Y since $X(Yh) - Y(Xh) = ([X, Y]h)$ vanishes at a critical point of h .

In local coordinates, this is just the matrix

$$H_h = \left(\frac{\partial^2 h}{\partial x^i \partial x^j} \right).$$

On a Riemannian manifold we can define the Hessian *everywhere* as

$$H_h(X, Y) = X \cdot (Y \cdot h)(m) - (\nabla_X Y) \cdot h(m).$$

This formula reduces to the formula given above *at a critical point* since the second term vanishes there, but it is easy to see that this expression is both tensorial and symmetric (from the fact that the Levi-Civita connection is torsion-free) everywhere.

²Since H_h is a symmetric bilinear form, we can choose a basis e_1, e_2, \dots, e_n of $T_m M$ such that

$$H_h(e_i, e_j) = \begin{cases} 0 & \text{if } i \neq j \\ \pm 1 \text{ or } 0 & \text{if } i = j \end{cases}.$$

The *nullity* is the dimension of the kernel ($= 0$ for a non-degenerate form).

The *index* is defined to be the number of (-1) 's. It is a routine linear algebra exercise to verify that these quantities are independent of the choice of basis.

the topology of M_c does not change as c varies unless c passes through a critical value.

The second idea of Morse theory is that the change in topology as c passes through a critical value is determined by the form of the Hessian at the critical value. In particular, the *index* of the Hessian determines completely the change in topology.

Hence, we should be able to compute topological invariants in terms of the critical points. For example, we might want to calculate the cohomology of the manifold. This will be our main goal, in fact.

First we must produce a canonical form for the Morse function near a critical point. This will allow us to make the necessary local computations. The canonical form is furnished by the following lemma, which is proved in all standard texts such as [3].

Lemma 1.1. *Let $h : M \rightarrow \mathbb{R}$ be a Morse function. Then around each critical point m_i there exists local coordinates (x^j) such that the Morse function has the form*

$$h(x) = \frac{1}{2} \sum \lambda_j (x^j)^2 + h(m_i).$$

Here the number of negative λ_j is equal to the index of the critical point.

Using this theorem, the standard approach to Morse theory [3] proves the following result using a CW-complex decomposition for M . This result (the so-called Poincare polynomial) relates the number M_p of critical points of index p to the p^{th} Betti number β_p of the real cohomology $H^*(M; \mathbb{R})$:

$$(1.1) \quad \sum_p M_p t^p - \sum_p \beta_p t^p = (1+t) \sum_p Q_p t^p.$$

Here the Q_p are nonnegative coefficients, and t is just some formal parameter that we can vary as necessary. The result is called the *Strong Morse Inequality*. A priori, it is clear that the strongest statement would result if we could find a Morse function h such that the Q_p all vanish. Such a Morse function is called *perfect*.

A weaker statement which is an easy consequence of equation 1.1 is called the *Weak Morse Inequality*:

$$(1.2) \quad M_p \geq \beta_p.$$

It will be our goal to prove these inequalities using a different method than the classic CW-complex construction.

2. WITTEN'S PERTURBED LAPLACIAN: IDEA

Equations 1.1 and 1.2 relate the critical points of a Morse function $h : M \rightarrow \mathbb{R}$ to the Betti numbers of the real cohomology $H^*(M; \mathbb{R})$. But it is well-known that the real cohomology is isomorphic to the de Rham cohomology $H_{\text{dR}}^*(M)$, so it makes sense to study the Morse inequalities using de Rham theory.

The de Rham cohomology $H_{\text{dR}}^p(M) = \ker(d|_{\Omega_p(M)}) / \text{im}(d|_{\Omega_{p-1}(M)})$ is constructed from the de Rham *complex*:

$$(2.1) \quad \Omega_0(M) \xrightarrow{d} \Omega_1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega_n(M).$$

Here the $\Omega_p(M)$ denote the p -forms in the exterior algebra ΛT^*M , and $\dim M = n$. Implicit in the classification of equation 2.1 as a *complex* we necessarily require $d^2 = 0$.

If we adorn the manifold M with a Riemannian structure, then the exterior algebra above can *also* be given a natural Clifford algebra structure, which defines a Dirac operator D [2]. In fact, we can calculate a formula for the Dirac operator in this case: $D = d + d^*$. A complex with a D defined by this formula is known as a *Dirac complex*.

Since our goal is to study the cohomology of this complex, we need to pick a representative p -form ω which generates each cohomology class $[\omega] \in H_{\text{dR}}(M)$. Hodge theory [2] provides a canonical way to choose this representative. In fact, what Hodge theory says is that the cohomology vector space is isomorphic to the vector space spanned by the so-called *harmonic* forms.

In other words, for every cohomology class we have *precisely one* harmonic form as the representative. As a reminder, a form ω is called *harmonic* if $D^2\omega = 0$ (in fact, this equation implies that $D\omega = d\omega = d^*\omega = 0$ separately).

So now it is only necessary to count *harmonic* forms. This simplifies our lives considerably since we only need to solve the harmonic equation $D^2\omega = 0$ to count the cohomology classes (find the Betti numbers).

2.1. Witten's perturbed complex. At this point we know that to count the Betti numbers it is only necessary to count the harmonic forms. But Equations 1.1 and 1.2 relate these Betti numbers to the number of critical points of a (as yet not mentioned) Morse function $h : M \rightarrow \mathbb{R}$. How is this achieved?

What Witten realized was that we should *not* study the usual Dirac operator D as found above (which has nothing to do with the Morse function), but rather a perturbed Dirac operator D_s which *does* have something to do with the Morse function h (the subscript s , which denotes a positive real number, will be explained shortly).

First, it is easy to see that multiplication of p -forms by e^{-sh} induces an (invertible) automorphism of the de Rham complex in equation 2.1 (in particular, we can just multiply by e^{sh} to undo the transformation). In order to make this induce an isomorphism on the cohomology $H_{\text{dR}}(M)$ (preserve the Betti numbers) we must also conjugate d by this same element, i.e. define $d_s \equiv e^{-sh}de^{sh} = d + s(dh \wedge)$ (so that $d_s\omega_s = e^{-sh}de^{sh}e^{-sh}\omega = (d\omega)_s$; i.e. closed forms remain closed, and exact forms remain exact).

So now we have a different Dirac complex with isomorphic cohomology, the so-called perturbed (or Witten) complex:

$$(2.2) \quad \Omega_0(M) \xrightarrow{d_s} \Omega_1(M) \xrightarrow{d_s} \dots \xrightarrow{d_s} \Omega_n(M).$$

The corresponding Dirac operator is just $D_s = d_s + d_s^*$ where $d_s^* = (d_s)^* = e^{sh}d^*e^{-sh} = d^* - s(dh \lrcorner)$ (\lrcorner denotes interior multiplication).³

The results of Hodge theory carry over verbatim (in fact Hodge theory is formulated for all Dirac complexes, not just the de Rham complex), and hence we merely need to count the harmonic forms for the perturbed Laplacian D_s^2 .

³Warning! The Dirac operator $D_s = D + s((dh \wedge) - (dh \lrcorner))$ is what is known as a *generalized* Dirac operator, since it is not actually the Dirac operator for the Clifford bundle. Instead it has an extra zero-order endomorphism. The elliptic theory used here is still valid for any extra piece up to first-order.

As we will see in the next subsection, the perturbed Laplacian takes the form in suitable local coordinates $D_s^2 = D^2 + s^2 |dh|^2 + s\mathfrak{H}_h$ where \mathfrak{H}_h is a bounded zero-order operator. Letting ω be harmonic ($D_s^2\omega = 0$), we have

$$\begin{aligned} 0 &= \langle (D^2 + s^2 |dh|^2 + s\mathfrak{H}_h)\omega, \omega \rangle \\ &= \langle D\omega, D\omega \rangle + s^2 |dh|^2 \langle \omega, \omega \rangle + s \langle \mathfrak{H}_h\omega, \omega \rangle . \end{aligned}$$

The first two terms are positive, and the last term is bounded by $s \|\mathfrak{H}_h\| \langle \omega, \omega \rangle$ since \mathfrak{H}_h is a bounded operator.

The important point here is that, by choosing s large, the middle term will dominate - as long as we keep ourselves away from any critical points (so $|dh|^2$ is bounded strictly away from zero). Hence, if we desire that $D_s^2\omega = 0$ (recall we are looking for harmonic forms) then in this region ω must be *vanishing* rapidly for large s to counteract the rapid growth of the middle term.

Near the critical points, however, is a different story. Since $|dh|^2 \sim 0$, the middle term does not contribute, and hence ω does *not* need to rapidly decrease there for large s . So the effect of taking $s \rightarrow \infty$ is to concentrate the harmonic forms around the critical points.

This is not a proof, but it does build our intuition as to how the harmonic forms (and hence the Betti numbers) are related to the critical points of a Morse function. What we'll show is that there is *at most* one harmonic form near each critical point which is not vanishing uniformly as $s \rightarrow \infty$, yielding the weak inequality 1.2 immediately. Our argument will be slightly more sophisticated and will actually yield the stronger inequality 1.1.

2.2. Local expression for the perturbed Laplacian. As mentioned, we must find a local expression for the perturbed Laplacian D_s^2 .⁴ The following calculation does not depend on whether M is compact or not (so, in particular, it applies to \mathbb{R}^n also). To this end, we first look for a local expression for the perturbed Dirac operator D_s :

$$D_s = d_s^* + d_s = d^* + d + s(dh \wedge -dh\lrcorner).$$

Pick a local orthonormal frame e_i . Since we are thinking of the exterior bundle ΛT^*M as a Clifford bundle, we have both a left *and* right Clifford multiplication.⁵ In fact, these actions commute (by associativity of the Clifford algebra, or by checking the formulas directly).

Now denote left multiplication of $e \in TM$ on $\omega \in \Lambda T^*M$ by $L_e\omega$, i.e.

$$L_e\omega \equiv e \cdot \omega.$$

⁴The following is almost completely (and shamelessly) stolen from [2]. It is included here for completeness.

⁵Usually we have only a left multiplication defined on the (left) Clifford module structure. But here the Clifford module is just the Clifford algebra itself, hence we have a natural right multiplication.

In fact, it is easily shown that left multiplication of $e \in TM$ on $\omega \in \Lambda T^*M$ is given by the formula

$$e \cdot \omega = e \wedge \omega + e\lrcorner\omega,$$

and similarly right multiplication is given by

$$\omega \cdot e = (-1)^{\text{rank } \omega} (e \wedge \omega - e\lrcorner\omega).$$

Define a (skew) right multiplication $R_e \omega$ by

$$R_e \omega \equiv (-1)^{\text{degree } \omega} \cdot e.$$

So then L_e and R_e anticommute, which is easily checked.

In this notation and framing we have

$$D_s \omega = D\omega + s(dh \wedge -dh \lrcorner) \omega = \sum_j L_{e_j} \nabla_j \omega + s(e_j \cdot h) R_{e_j} \omega.$$

Squaring this gives

$$D_s^2 = \sum_{i,j} (L_{e_j} \nabla_j \omega + s(e_j \cdot h) R_{e_j} \omega) (L_{e_i} \nabla_i \omega + s(e_i \cdot h) R_{e_i} \omega),$$

which is just

$$(2.3) \quad D_s^2 = D^2 + s^2 |dh|^2 + s \mathfrak{H}_h.$$

Here \mathfrak{H}_h is a zero-order endomorphism given by the formula

$$\mathfrak{H}_h = \sum_{i,j} H_h(e_i, e_j) L_{e_i} R_{e_j}$$

where H_h is the Hessian of h . The above formulas can be obtained from the facts that L and R anticommute and that $\sum_{i,j} R_{e_i} R_{e_j} = \sum_i \delta_{ij}$.

Now, if we are in flat (Euclidean) space equipped with Morse coordinates as in lemma 1.1 (so we have a single critical point at the origin), then equation 2.3 reduces to

$$(2.4) \quad D_s^2 = \sum_{\dim M} \left(- \left(\frac{\partial}{\partial x^j} \right)^2 + s^2 (\lambda_j x^j)^2 + s \lambda_j Z_j \right)$$

where $Z_j \equiv [dx^j \lrcorner, dx^j \wedge]$ takes the value +1 on a basis element $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ if $j \in \{i_1, \dots, i_k\}$ and -1 if $j \notin \{i_1, \dots, i_k\}$.

2.3. The spectrum of the perturbed Laplacian on \mathbb{R}^n . It remains to find the spectrum of D_s^2 so that we can find the zero eigenvalues. This is easy for the Euclidean plane \mathbb{R}^n from equation 2.4. Let $Y_j = - \left(\frac{\partial}{\partial x^j} \right)^2 + s^2 (\lambda_j x^j)^2$, so then

$$D_s^2 = \sum_{\dim M} (Y_j + s \lambda_j Z_j).$$

In terms of a basis $\{dx^{i_1} \wedge \cdots \wedge dx^{i_k}\}$ for the exterior bundle, Z_j is just a constant diagonal matrix (made up of ± 1 's). Thus, all Z and Y operators commute, and hence these operators can be simultaneously diagonalized.

If we limit ourselves to $L^2(\mathbb{R}^n)$, then the spectrum of Y_j is well known from the study of the harmonic oscillator, and it is just

$$s |\lambda_j| (1 + 2p_j) \quad p_j = 0, 1, 2, \dots$$

Furthermore the spectrum from Z_j is just $q_j = -1, 1$, which follows from the comment after equation 2.4.

Putting these results together, the spectrum for equation 2.4 is given by

$$(2.5) \quad s \sum_j (|\lambda_j| (1 + 2p_j) + \lambda_j q_j).$$

This only vanishes if $p_j = 0$ and if q_j is $+1$ if λ_j is negative and -1 if λ_j is positive.⁶

These conditions determine a *single* eigenform ω_0 completely as (normalized)

$$(2.6) \quad \omega_0 = \left[s^{n/2} \pi^{-n/4} \prod_j |\lambda_j|^{1/2} \right] \exp \left(-s \sum_j |\lambda_j| (x^j)^2 / 2 \right) dx^{i_1} \wedge \dots \wedge dx^{i_p},$$

where $\{\lambda_{i_1}, \dots, \lambda_{i_p}\}$ are the negative coefficients in the Morse lemma 1.1. This is the only eigenform of D_s^2 with vanishing eigenvalue (the rest are positive).

3. THE PROOF

We are ready to attack the Morse inequalities directly. Let M be a compact manifold with some Riemannian metric (we'll construct a convenient one later), and let $h : M \rightarrow \mathbb{R}$ be a Morse function.

Consider the following construction from functional calculus [2]: let $\phi : \mathbb{R} \rightarrow \mathbb{R}^+$ be a smooth rapidly decreasing function such that $\phi(0) = 1$. We know from elliptic theory that the eigenvalues $0 \leq \lambda_0 \leq \lambda_1 \leq \dots$ of D_s^2 are nonnegative, discrete, do not have an accumulation point (in particular only finitely many λ_i 's may be equal to one another), and tend to infinity. Furthermore, the set of (orthogonal) eigenforms ω_i span the entire Clifford bundle (in this case the exterior bundle). So if $\omega \in \Lambda T^*M$ then we can decompose $\omega = \sum_i \alpha_i \omega_i$ for some coefficients α_i .

Now construct an operator, denoted $\phi(D_s^2)$, by $\phi(D_s^2)\omega \equiv \sum_i \alpha_i \phi(\lambda_i) \omega_i$. It follows from functional calculus [2] that this operator is bounded, trace-class, and smoothing (has a smoothing kernel).

With this in mind, consider the quantity

$$(3.1) \quad \mu_p \equiv \text{tr}(\phi(D_s^2 |_{\Omega_p(M)})).$$

Here we have named it μ_p in anticipation that, in the limit of large s , this will just become the number of critical points of h with Morse index p , i.e. $\mu_p \xrightarrow{s \rightarrow \infty} M_p$. The bulk of our work will be in proving this. Granting this for now, the following proposition then proves the Morse inequality 1.1:

Proposition 3.1. *For each $s \geq 0$ we have the relation $\sum_p \mu_p t^p - \sum_p \beta_p t^p = (1+t) \sum_p Q_p t^p$ where Q_p is a nonnegative number.*

Proof. We have shown that, for each $0 \leq p \leq \dim M - 1$, the Betti number β_p is the dimension of the kernel of $D_s^2 |_{\Omega_p(M)}$ (the harmonic forms). We also know from elliptic theory, as mentioned previously, that the spectrum of $D_s^2 |_{\Omega_p(M)}$ is nonnegative, discrete, and does not accumulate. Thus for each s we can pick a smooth cutoff function $\tilde{\phi}_s : \mathbb{R} \rightarrow \mathbb{R}^+$ such that $\tilde{\phi}_s(0) = 1$ and $\tilde{\phi}_s(\lambda) = 0$ for any nonzero eigenvalue λ of D_s^2 . These observations together imply that $\beta_p = \text{tr}(\tilde{\phi}_s(D_s^2 |_{\Omega_p(M)}))$.

We are free to choose $\tilde{\phi}_s \leq \phi$, so then

$$(\phi - \tilde{\phi}_s)(\lambda) = \lambda(\psi_s(\lambda))^2$$

where ψ_s is a smooth, rapidly-decreasing function.

Using this expression, we have

$$(\mu_p - \beta_p)t^p = \text{tr}((\phi - \tilde{\phi}_s)(D_s^2 |_{\Omega_p(M)}))t^p = \text{tr}(D_s^2(\psi_s(D_s^2 |_{\Omega_p(M)}))^2)t^p.$$

⁶It is not hard to show (using local submersion theorem) that *generically* (i.e. for small perturbations of h) the only way that equation 2.5 vanishes is under these conditions.

But $D_s^2 = d_s d_s^* + d_s^* d_s$, and furthermore

$$\begin{aligned} \operatorname{tr} (d_s d_s^* (\psi_s(D_s^2 |_{\Omega_p(M)}))^2) &= \operatorname{tr} (\psi_s(D_s^2 |_{\Omega_p(M)}) d_s d_s^* \psi_s(D_s^2 |_{\Omega_p(M)})) \\ &\stackrel{\text{cyclicity}}{=} \operatorname{tr} (d_s^* (\psi_s(D_s^2 |_{\Omega_p(M)}))^2 d_s) \\ &\stackrel{\text{commutativity}}{=} \operatorname{tr} (d_s^* d_s (\psi_s(D_s^2 |_{\Omega_{p-1}(M)}))^2) \end{aligned} .$$

The first equality follows from the fact that $\psi_s(D_s^2)$ is trace-class and $d_s d_s^* \psi_s(D_s^2)$ is bounded, and the second equality follows from the fact that both $\psi_s(D_s^2) d_s$ and $d_s^* \psi_s(D_s^2)$ are bounded and Hilbert-Schmidt [2]. The third equality follows from the fact that d_s commutes with $(\psi_s(D_s^2 |_{\Omega_p(M)}))^2$, at a cost of lowering $p \rightarrow p - 1$.

Collecting these results, we see that

$$(\mu_p - \beta_p) t^p = (\operatorname{tr} (d_s^* d_s (\psi_s(D_s^2 |_{\Omega_p(M)}))^2) + \operatorname{tr} (d_s^* d_s (\psi_s(D_s^2 |_{\Omega_{p-1}(M)}))^2)) t^p .$$

Summing over each dimension and telescoping the series gives

$$\sum_p (\mu_p - \beta_p) t^p = (t + 1) \sum_p \operatorname{tr} (d_s^* d_s (\psi_s(D_s^2 |_{\Omega_p(M)}))^2) t^p .$$

But we notice that

$$\operatorname{tr} (d_s^* d_s (\psi_s(D_s^2 |_{\Omega_p(M)}))^2) = \operatorname{tr} (A_s^* A_s) \geq 0$$

where $A_s \equiv d_s \psi_s(D_s^2 |_{\Omega_p(M)})$. This proves the inequality. \square

We will busy ourselves with the rest of the proof showing that $\mu_p \xrightarrow{s \rightarrow \infty} M_p$.

One trick we need is to choose ϕ to be an even function, which allows us to perform calculations using $\phi(D_s)$ instead of $\phi(D_s^2)$ when necessary. This will sometimes be helpful, and by abuse of notation we will often switch between the two without mention.

Now we can construct a Riemannian structure on M as follows. Since we have a Morse function $h : M \rightarrow \mathbb{R}$ on our compact manifold, we are dealing with a finite number of isolated critical points m_i . Using the Morse lemma 1.1, pick a neighborhood around each critical point and *define* the metric to be flat with respect to the Morse coordinates on this neighborhood. Then, using a partition of unity, we can easily patch together this metric with an arbitrary metric away from the critical points.

Having now a Riemannian structure on M (locally flat near each critical point), choose a number ρ small enough such that around each critical point we can assure that the metric is flat at least to a distance 4ρ . Now choose a different partition of unity $\{A, B_i\}$ such that for each critical point m_i we have that B_i is supported in a 3ρ -neighborhood of m_i and is equal to unity on a 2ρ -neighborhood (so the manifold is flat on $\operatorname{supp} B_i$). The function A is supported outside a distance 2ρ (away) from every critical point and is unity at a distance of 3ρ . Then we have that, for every point m on the manifold M

$$A(m) + \sum_i B_i(m) = 1.$$

Denote the smoothing kernel of $\phi(D_s^2)$ by $k_s(m_1, m_2)$ where $m_1, m_2 \in M$. Then, by the general theory of smoothing operators [2], the trace in equation 3.1 can be written as an integral:

$$\mu_p \equiv \operatorname{tr} (\phi(D_s^2 |_{\Omega_p(M)})) = \int \operatorname{tr} (k_s^p(m, m)) \operatorname{vol}(m).$$

Plugging the partition of unity into this integral expression gives

$$(3.2) \quad \int \operatorname{tr} k_s^p(m, m) \operatorname{vol}(m) = \int \operatorname{tr} k_s^p(m, m) A(m) \operatorname{vol}(m) + \sum_i \int \operatorname{tr} k_s^p(m, m) B_i(m) \operatorname{vol}(m)$$

First, we wish to show that, in the limit of $s \rightarrow \infty$, the region away from the critical points does not contribute to the trace (the first term in the above equation vanishes).

Then we will calculate the contribution from the critical points by calculating the sum in the second term. What we will see is that, in the limit, each critical point of index p contributes a 1, and other critical points contribute nothing.

Taken together, these results show indeed that $\mu_p \xrightarrow{s \rightarrow \infty} M_p$.

3.1. Away from the critical points. As advertised, first we need to show that the region on M away from the critical points contributes nothing to the trace. So our goal is to show that in equation 3.2 the term

$$(3.3) \quad \int \operatorname{tr} k_s^p(m, m) A(m) \operatorname{vol}(m)$$

vanishes as $s \rightarrow \infty$.

But this integral is cut off to a region that supports A (supported *away* from the critical points). So by Lebesgue's Dominated Convergence Theorem, the following proposition shows that the contribution to the trace away from the critical points vanishes as $s \rightarrow \infty$.

Proposition 3.2. *Outside of a 2ρ -neighborhood of the critical points m_i , the smoothing kernel of $\phi(D_s^2)$ tends uniformly to zero as $s \rightarrow \infty$.*

Proof. We are considering the case that M is compact, hence away from the critical points we obtain a constant C such that $|dh| \geq C > 0$ (note that this same argument will apply to the open manifold \mathbb{R}^n equipped with Morse coordinates as in lemma 1.1 since we have only a single critical point in this case).

Let $\omega \in L^2(M)$ be supported outside a 2ρ -neighborhood of the critical points. Then using equation 2.3, and by the above considerations, we have for large enough s the bound

$$\langle D_s^2 \omega, \omega \rangle \geq \frac{1}{4} C^2 s^2 \|\omega\|_{L^2}^2.$$

⁷ This equation implies that D_s^2 is a (formally self-adjoint) positive operator away from the critical points, hence its square root operator D_s is positive and bounded below by $\frac{1}{2}Cs$.

By the spectral theorem we then see that the L^2 operator norm of $\phi(D_s)$ is bounded above by

$$c(s) = \sup \left\{ |\phi(\lambda)| : \lambda \geq \frac{1}{2}Cs \right\},$$

where $c(s)$ is vanishing rapidly as $s \rightarrow \infty$. In other words, for every ω supported away from the critical points we have

$$(3.4) \quad \|\phi(D_s)\omega\|_{L^2} \leq c(s) \|\omega\|_{L^2}.$$

⁷Notice that the same choice of s works for all such ω since in equation 2.3 the operator D^2 is positive and the operator \mathfrak{H}_h is bounded.

This is nearly what we want. What we really want to show is that there exists a $c_1(s)$ which rapidly vanishes as $s \rightarrow \infty$ such that

$$(3.5) \quad \|\phi(D_s)\omega\|_{L^\infty} \leq c_1(s) \|\omega\|_{L^1}.$$

Then, using the fact that L^1 is dual to L^∞ , this shows that the smoothing kernel of $\phi(D_s)$ vanishes in the L^∞ norm as $s \rightarrow \infty$, which finishes the proof.

So it remains to show that equation 3.4 leads to equation 3.5. Let W^k denote the k th Sobolev space, and let the following norms be Sobolev norms. Garding's inequality says

$$\|\omega\|_{k+1} \leq C_k(\|\omega\|_k + \|D_s\omega\|_k).$$

Using this, it can be shown [4] that the operator $(1 + D_s^2)^{-1}$ is bounded as an operator from W^p to W^{p+2} (the bound being a polynomial in s), so repeated application shows that $(1 + D_s^2)^{-k}$ is bounded from $W^0 \equiv L^2$ to W^{2k} . Now the Sobolev embedding theorem says that if $2k > \frac{n}{2}$ then $W^{2k} \subset L^\infty$.

Summarizing, we see that if $2k > \frac{n}{2}$ then $(1 + D_s^2)^{-k}$ is a bounded (by a polynomial in s) operator from L^2 to L^∞ . This gives

$$\begin{aligned} \|\phi(D_s)\omega\|_{L^\infty} &= \|(1 + D_s^2)^{-k}(1 + D_s^2)^{2k}(1 + D_s^2)^{-k}\phi(D_s)\omega\|_{L^\infty} \\ &\leq \text{poly}(s) \|(1 + D_s^2)^{2k}(1 + D_s^2)^{-k}\phi(D_s)\omega\|_{L^2}. \end{aligned}$$

Now every factor in the above equation commutes since ϕ is an even function (so $\phi(D_s)$ can be thought of as $\phi(D_s^2)$), so this is just

$$\begin{aligned} &= \text{poly}(s) \|(1 + D_s^2)^{2k}\phi(D_s)(1 + D_s^2)^{-k}\omega\|_{L^2} \\ &= \text{poly}(s) \|\tilde{\phi}(D_s)(1 + D_s^2)^{-k}\omega\|_{L^2} \end{aligned}$$

where $\tilde{\phi}$ is also an *even*, rapidly decreasing function.

We can use duality again to finish the proof. Since $(1 + D_s^2)^{-k} : L^2 \rightarrow L^\infty$ is bounded, we have that $((1 + D_s^2)^{-k})^* : (L^\infty)^* = L^1 \rightarrow (L^2)^* = L^2$ is bounded. But $(1 + D_s^2)^{-k}$ is self-adjoint, so we really have that $(1 + D_s^2)^{-k} : L^1 \rightarrow L^2$ is also bounded (again, by a polynomial in s).

Applying equation 3.4 and this last bound, we have that

$$\begin{aligned} &= \text{poly}(s) \|\tilde{\phi}(D_s)(1 + D_s^2)^{-k}\omega\|_{L^2} \\ &\leq \text{poly}(s)c(s) \|(1 + D_s^2)^{-k}\omega\|_{L^2}, \\ &\leq c_1(s) \|\omega\|_{L^1} \end{aligned}$$

where $c_1(s) = \text{poly}(s)\text{poly}(s)c(s)$ is vanishing as $s \rightarrow \infty$. This proves equation 3.5, which proves the proposition. \square

3.2. Contribution from the critical points. The only remaining task is to calculate the sum around the critical points (the last term) in equation 3.2. So consider the expression around a single critical point

$$(3.6) \quad \int \text{tr } k_s^p(m, m)B(m) \text{vol}(m)$$

where we have dropped the index on B . Recall that B is supported in a 3ρ -neighborhood of the critical point, and is unity on a 2ρ -neighborhood.

But we know that on $\text{supp } B$ the manifold M is Euclidean, hence we obtain⁸

$$\int_M \text{tr } k_s^p(m, m) B(m) \text{vol}(m) = \int_{\mathbb{R}^n} \text{tr } k_s^p(m, m) B(m) \text{vol}(m).$$

Here the subscript M on the LHS was left implicit before, but now we explicitly decorate the integrals so as to contrast it with the RHS.

Now, on \mathbb{R}^n (choosing the standard metric) we can play the same game that we played on M . First, we can give \mathbb{R}^n a Morse function by simply defining $h : \mathbb{R}^n \rightarrow \mathbb{R}^+$ as in lemma 1.1. Then manifestly we have only one critical point (at the origin). Define cutoff functions A and B on \mathbb{R}^n as in equation 3.2. Then, rearranging equation 3.2 (but applied to \mathbb{R}^n), we find the expression

$$\int_{\mathbb{R}^n} \text{tr } k_s^p(m, m) B(m) \text{vol}(m) = \int_{\mathbb{R}^n} \text{tr } k_s^p(m, m) \text{vol}(m) - \int_{\mathbb{R}^n} \text{tr } k_s^p(m, m) A(m) \text{vol}(m).$$

But the last term on the RHS vanishes as $s \rightarrow \infty$ by proposition 3.2 (which also applies to \mathbb{R}^n in this situation, as discussed). Hence we have that

$$\int_{\mathbb{R}^n} \text{tr } (k_s^p(m, m)) B(m) \text{vol}(m) \xrightarrow{s \rightarrow \infty} \int_{\mathbb{R}^n} \text{tr } k_s^p(m, m) \text{vol}(m).$$

Again, using the general theory of smoothing operators (this time on an open manifold), we have that the RHS is just

$$\int_{\mathbb{R}^n} \text{tr } k_s^p(m, m) \text{vol}(m) = \text{tr}_{L^2(\mathbb{R}^n)} (\phi(D_s^2 |_{\Omega_p(\mathbb{R}^n)})).$$

The trace on the right is just the sum over $\phi(\text{eigenvalues})$ of D_s^2 (*restricted to p -forms*) in Euclidean space. The analysis in \mathbb{R}^n was already performed in subsection 2.3. It follows from equation 2.5 that, in the limit $s \rightarrow \infty$, this trace just counts the number of zero eigenvalues of $D_s^2 |_{\Omega_p(M)}$, since the nonzero eigenvalues diverge with s (and hence are killed by ϕ). From equation 2.6 we have only 1 “zero” eigenform for D_s^2 , and it is a p -form if the critical point m_i has index p . In other words, the RHS is just 1 if the critical point m_i has index p , and is zero otherwise. Thus, the trace in equation 3.1 does indeed approach M_p as $s \rightarrow \infty$. This proves the theorem.

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MATHEMATICS DEPARTMENT, UNIVERSITY OF TEXAS AT AUSTIN
E-mail address: `stirling@math.utexas.edu`

⁸This equality occurs since the smoothing kernel $k_s(m_1, m_2)$ on $\text{supp } B$ is only determined by $\phi(D_s^2)$ on $\text{supp } B$ (where M is Euclidean).