THE ATIYAH-BOTT FORMULATION OF THE LEFSCHETZ THEOREM

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ABSTRACT. In this paper we review a formula due to Atiyah and Bott for the Lefschetz number of a map $f : M \to M$ on a manifold M. Their insight was to study the Lefschetz number using Dirac complexes over M and standard techniques from elliptic operator theory. First we give a general definition for the Lefschetz number in terms of cohomology, then we present the main formula derived from it by Atiyah and Bott. Before proving it, however, we consider a couple of examples. First we show that this formula reduces to the usual definition of the Lefschetz number (from differential topology) when the Dirac complex is just the usual de Rham complex. Then we compute for $M = \mathbb{CP}^2$ the Lefschetz number for a map homotopic to the identity. Finally, we proceed to review the proof as presented in [1].

1. INTRODUCTION

In elementary differential topology we study the de Rham complex over a manifold M. This means that we have a vector bundle $\Lambda^* T^* M$ over M (the exterior algebra) which decomposes into subbundles $\Lambda^* T^* M = \oplus \Lambda^i T^* M$. The smooth sections of $\Lambda^i T^* M$ are denoted $C^\infty(\Lambda^i T^* M) \equiv \Omega^i(M)$, and we have a differential operator $d : \Omega^i(M) \to \Omega^{i+1}(M)$ such that $d^2 = 0$. More succinctly, we have the complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0.$$

In this context we usually study the Lefschetz number $L(f)$ associated to a smooth map $f : M \to M$ by considering the local behavior of f near its fixed points. In terms of the fixed points p, the Lefschetz number is

$$L(f) = \sum_p \text{sign}\ \det(1 - T_p f). \tag{1.1}$$

It can be shown [2] using other methods, however, that this formula is a consequence of a more general formula involving the cohomology of M. Indeed, the map $f : M \to M$ induces an endomorphism $f^* : H^i(M) \to H^i(M)$ on the cohomology groups of M. In terms of this induced endomorphism, the general formula for the Lefschetz number is

$$L(f) = \sum_i (-1)^i \,\text{tr}\,(f^* \text{ on } H^i(M)). \tag{1.2}$$

In the next section we will extend our sights beyond the de Rham complex and consider more general Dirac complexes. In this case we also have a cohomology structure (different than the usual de Rham cohomology) defined on the Dirac complex, and under the right conditions the map $f : M \to M$ induces an endomorphism on this cohomology.

Using *this* cohomology structure in equation 1.2, we will prove a generalization of equation 1.1 due to Atiyah and Bott [3].

2. Extension to Dirac Complexes

To generalize the de Rham complex discussed in the last section we consider complexes which arise naturally in the study of elliptic theory. Let M be an n-dimensional compact oriented Riemannian manifold, and let S be a Clifford bundle over M. Suppose that S decomposes into a direct sum of vector subbundles $S = \oplus S^i$, and furthermore assume that these subbundles arrange themselves into a complex with differential operators $d_i : C^\infty(S^i) \to C^\infty(S^{i+1})$. Calling this structure a *complex* implicitly assumes that $d_{i+1} d_i = 0$. So this complex looks like

$$C^\infty(S^o) \xrightarrow{d_0} C^\infty(S^1) \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} C^\infty(S^k). \tag{2.1}$$

We say that this complex is a *Dirac complex* if the Dirac operator D associated to the Clifford bundle S is just given by the formula $D = d + d*$. It is easy to see that the de Rham complex discussed in section 1 is an example of a Dirac complex.

Since our main goal is to study the Lefschetz theorem, consider a smooth map $f : M \to M$. This map naturally lifts to a map $f^* : C^\infty(S) \to C^\infty(f^*S)$, which is almost what we want. We *must assume* that there also exists a smooth bundle map $\zeta : f^*S \to S$, so then we have a map

$$F = \zeta \circ f^* : C^\infty(S) \to C^\infty(S).$$

Note that the necessary bundle map exists (almost without need of mention) for the case of the de Rham complex, and it is just $\zeta = \Lambda^* T^* f$.

What we *really* want to consider, however, is not just an induced map of sections, but rather an induced map of the *cohomology* created from these sections in the complex 2.1. So we require one further assumption, which is just the formula $Fd = dF$, i.e. that F is a *chain map*, or a *geometric endomorphism.* It is easy to see that this requirement induces a well-defined endomorphism F^* on the cohomology groups $H^i(S) \equiv \ker d_i / \operatorname{im} d_{i-1}$. So taking equation 1.2 as our model, we define

$$L(f, \zeta) = \sum_i (-1)^i \operatorname{tr}(F^* \text{ on } H^i(S)). \tag{2.2}$$

This was the first insight due to Atiyah and Bott. The main goal is to derive a generalization of equation 1.1. This requires a steady hand in the use of elliptic theory, but following their lead is surprisingly simple. For now, let us content ourselves with the main result.

2.1. Main Result.

Let $f : M \to M$ be a Lefschetz map [2] (i.e. the fixed points p are *simple*, which means that $\det(1 - T_p f) \neq 0$). Then the Lefschetz number is given as a sum over the fixed points by

$$L(f, \zeta) = \sum_p \sum_i \frac{(-1)^i \operatorname{tr}(\zeta(p) \mid_{S^i})}{\mid \det(1 - T_p f) \mid}. \tag{2.3}$$

Before we prove this, let us consider a few examples.

3. EXAMPLE: DE RHAM COMPLEX

First we should verify that equation 2.3 is indeed a generalization of equation 1.1. This is relatively simple. We only need to calculate $\sum_i (-1)^i \operatorname{tr}(\zeta(p) \mid_{S^i})$ for the case that $\zeta(p) \mid_{S^i} = \zeta(p) \mid_{\Omega^i(M)} = \Lambda^i T^* f$.

Denote by A the linear transformation $T^*f : T^*M \to T^*M$. Then it is straightforward to verify (by choosing an orthonormal basis of $\Lambda^i A$) that $\operatorname{tr}(\Lambda^i A) = s_i(\lambda_1, ..., \lambda_n)$ where the λ_i denote the eigenvalues of A and s_i is the i^{th} symmetric function in n variables. Then it is straightforward to see that

$$\sum_i (-1)^i \operatorname{tr}(\Lambda^i A) = \det(1 - A).$$

Plugging this into equation 2.3 we find that

$$L(f) = \sum_p \frac{\det(1 - T_p^* f)}{\mid \det(1 - T_p f) \mid} = \sum_p \text{sign } \det(1 - T_p f)$$

where the last equality follows since in finite dimensions a map and its dual share the same eigenvalues. But this is just equation 1.1, as desired.

4. EXAMPLE: $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ HOMOTOPIC TO THE IDENTITY

For a more explicit example, consider the space $\mathbb{CP}^2$ which is just the space $\mathbb{C}^3$ with complex lines collapsed to a point. Define the map $A : \mathbb{C}^3 \to \mathbb{C}^3$ as

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Then clearly A induces a map $f : \mathbb{CP}^2 \to \mathbb{CP}^2$. It is trivial to see that f is homotopic to the identity (which we will use later). What are the fixed points of f? Since $\mathbb{CP}^2$ is just the space of lines, the fixed points correspond precisely to the eigenvectors v_λ of A. These are just

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Consider only the first fixed point v_2 (the other fixed points will be similar). Then in $\mathbb{CP}^2$ this is just the point $[1, 0, 0]$. We pick a coordinate system around this point $[1, x, y]$. Then

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3x \\ 4y \end{pmatrix} \sim \begin{bmatrix} 1 \\ 3x/2 \\ 2y \end{bmatrix} \text{ in } \mathbb{CP}^2.$$

In other words, in $\mathbb{CP}^2$ around the fixed point $[1, 0, 0]$, $f([1, x, y]) = [1, 3x/2, 2y]$. So the derivative at the fixed point is just

$$T_{p=v_2} f = \begin{pmatrix} 3/2 & 0 \\ 0 & 2 \end{pmatrix},$$

and similarly for the other fixed points

$$\begin{aligned} T_{p=v_3}f &= \begin{pmatrix} 2/3 & 0 \\ 0 & 4/3 \end{pmatrix} \\ T_{p=v_4}f &= \begin{pmatrix} 1/2 & 0 \\ 0 & 3/4 \end{pmatrix} \end{aligned}. \tag{4.1}$$

4.1. $\mathbb{CP}^2$ **as an $\mathbb{R}$-manifold; the de Rham complex.** Now let us think of $\mathbb{CP}^2$ as a real manifold, and let the Dirac complex be just the ordinary de Rham complex. Then equation 1.1 is the formula for the Lefschetz number in this case, so we need to compute the values for $\det_{\mathbb{R}}(1 - T_p f)$.

In this case each complex coordinate splits into two real coordinates, and hence each entry in equation 4.1 should really be considered as a 2 × 2 block. So, for example, at the fixed point $p = v_2$ we have

$$\det_{\mathbb{R}}(1 - T_{p=v_2}f) = \det \begin{pmatrix} 1-3/2 & 0 & 0 & 0 \\ 0 & 1-3/2 & 0 & 0 \\ 0 & 0 & 1-2 & 0 \\ 0 & 0 & 0 & 1-2 \end{pmatrix} = 1/4.$$

So sgn $\det_{\mathbb{R}}(1 - T_{p=v_2}f) = +1$. An identical computation shows that the other two fixed points also contribute a +1, hence the Lefschetz number for this map $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ is just

$$L(f) = 3.$$

This is fortunate, since we already know from differential topology that the Lefschetz number of a map homotopic to the identity is just the Euler characteristic! We'll come back to this comment in a bit.

4.2. $\mathbb{CP}^2$ **as a Kähler manifold; the Dolbeault complex.** $\mathbb{CP}^2$ also admits a complex structure, and in fact can be a Kähler manifold[1]. In this case we take a different Dirac complex which is called the Dolbeault complex [1]. Analogously to section 3 it is possible to reduce equation 2.3 to a form that is more convenient for our calculations. The details are outlined elsewhere, and the result is just

$$L(f) = \sum_p \frac{1}{\det_{\mathbb{C}}(1 - T_p f)}. \tag{4.2}$$

So let us compute a simple example. Using the same map $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ considered in the previous subsection, we see from equation 4.1 that

$$\det_{\mathbb{C}}(1 - T_p f) = \det_{\mathbb{C}} \begin{pmatrix} 1-3/2 & 0 \\ 0 & 1-2 \end{pmatrix} = 1/2.$$

Similarly, for the other fixed points we find

$$\begin{aligned} \det_{\mathbb{C}}(1 - T_p f)_{p=v_3} &= -1/9 \\ \det_{\mathbb{C}}(1 - T_p f)_{p=v_4} &= 1/8 \end{aligned}.$$

Plugging this into equation 4.2 yields the result for the Dolbeault complex

$$L(f) = 1.$$

This number is usually called the Todd genus.

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[1] This and the subsequent subsection will be far less self-contained than the remainder of the paper. For introductory details consult [1].

4.3. $\mathbb{CP}^2$ **and the Signature operator.** Another complex that is often considered has as its Dirac operator the so-called Signature operator D^+. It is a fact that every oriented Riemannian manifold of even dimension $(2n)$ admits a Signature operator (and a corresponding length-2 complex). This material is explained in detail in [4].

The important point here is that we *can* show that the Index of D^+ is a topological invariant of the manifold. Hence it makes sense to give it a special name: the Signature of M.

Now, as explained in the next section for general Dirac operators, if we pick a map $f : M \to M$ which is homotopic to the identity and calculate its Lefschetz number, then this *also* happens to be the Index of D^+. Hence we can calculate the Signature of M by calculating the Lefschetz number of an appropriate map homotopic to the identity (it so happens that we also require f to be an isometry in order to produce a well-defined geometric endomorphism of this Dirac complex).

As is becoming a recurring theme here, equation 2.3 can be specialized to this complex. It turns out that the Signature is given as a sum over fixed points by

$$\text{Sign}(f) = L(f) = \sum_p i^{-n+1} \prod_{k=1}^{n} \frac{e^{i\theta_k} + e^{-i\theta_k}}{e^{i\theta_k} - e^{-i\theta_k}} \tag{4.3}$$

where again $\dim(M)=2n$. Here the θ_k are the angles of rotation of 2-dimensional (1-d complex) subspaces of the tangent space (at the fixed point p) induced by the isometry f (an isometry of a manifold induces an isometry of the tangent space at a fixed point, hence only rotations through angles are allowed).

So now let's calculate the signature for the isometry $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ induced from the map $A : \mathbb{C}^3 \to \mathbb{C}^3$ where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{i2\theta} \end{pmatrix}.$$

Here θ is just some arbitrary angle. Then a calculation identical to that spelled out at the beginning of this entire section yields 3 fixed points of f (eigenspaces of A) with derivatives in $\mathbb{CP}^2$

$$\begin{aligned} T_{p1}f &= \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i2\theta} \end{pmatrix} \\ T_{p2}f &= \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \\ T_{p3}f &= \begin{pmatrix} e^{-i2\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \end{aligned}.$$

These derivatives spell out explicitly what the θ_k are in equation 4.3. For example, at the first fixed point $p1$ we have one 2-dimensional subspace which rotates through an angle θ, and another which rotates through an angle 2θ.

So letting $\lambda = e^{i\theta}$ and plugging this into equation 4.3 yields (after a fair amount of algebra and some serendipitous cancellations)

$$\text{Sign}(f) = L(f) = 1.$$

We could go further and try to compute other topological quantities, such as the $\widehat{A}$-genus[2], but for now let us content ourselves with these.

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[2] We intended to do this, but time did not permit.

4.4. **Maps homotopic to the identity and the Index of D.** In the previous subsections we alluded to the link between the Lefschetz number, the Euler characteristic, and the Index of a Dirac operator. Let us spell this out for completeness.

Let $f : M \to M$ be homotopic to the identity. Then, in particular, f induces the identity map on the cohomology of the Dirac complex, hence equation 2.2 is just the alternating sum of the Betti numbers of the complex.

Just like in the case of the usual de Rham complex, we define this alternating sum to be the Euler characteristic of the complex.

Now, switching gears for a moment, suppose we have a graded Clifford bundle, i.e. we have a decomposition $S = S^+ \oplus S^-$. Then necessarily the Dirac operator D anticommutes with the grading operator, so it sends sections of $S_\pm$ to sections of $S_\mp$. In other words, we have a separate induced Dirac complex over M:

$$0 \to C^\infty(S_+) \overset{D|_{S_+}=D_+}{\to} C^\infty(S_-) \to 0.$$

We call the Euler characteristic of this complex the *index of D*.

Observe now that the Euler characteristic of this length-2 "$\mathbb{Z}_2$-graded Clifford complex" is actually the same as the Euler characteristic of the de Rham complex (or the Dolbeault complex), which are graded by $\mathbb{Z}$. The induced $\mathbb{Z}_2$ grading is given by $(-1)^i$. For example, odd (even)-powered differential forms in the de Rham complex are odd (even).

Hence, computing the Lefschetz number for a map homotopic to the identity is the same as computing the index of D. So in the examples above we were just computing indices of Dirac operators.

5. The Heat Equation Proof

It remains to prove Atiyah and Bott's formula 2.3. We start from equation 2.2, and we use Hodge theory to identify the vector space $H^i(S)$ with the vector space $\mathfrak{H}^i(S)$ spanned by the harmonic sections of S^i. Here we follow closely [1].[3]

Here we require some more results from Hodge theory to make a complete analysis of the cohomology (for details, see [1]). In particular there exists an orthogonal projection operator $P^i : C^\infty(S_i) \to \mathfrak{H}^i(S)$ which sends arbitrary smooth sections to harmonic sections (in fact the projection operator is defined for $L^2(S_i)$ sections, but here we restrict our attention to smooth sections). Furthermore this projection operator can be written as a smooth function $\phi(D^2)$ where $\phi(0) = 1$ and $\phi(\lambda) = 0$ for all other λ in the spectrum of D^2 (this follows since D^2 has discrete spectrum). So in particular P^i is a smoothing operator (i.e. P^i has a smoothing *kernel*).

From this point of view we rewrite the trace in equation 2.2 as

$$\operatorname{tr}(F^* \text{ on } H^i(S)) = \operatorname{tr}(FP^i).$$

Now we wish to study the expression on the RHS. In some sense we can show that P^i looks like $e^{-tD^2|_{S^i}}$ in the limit of large t. More precisely, we have the following claim:

Claim 5.1. In the limit as $t \to \infty$ the smoothing kernel of $e^{-tD^2|_{S^i}}$ approaches the smoothing kernel of P^i in the C^∞ topology.

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[3] We have added a few calculations, and summarized several others. The reader is advised to consult [1] for more details.

Proof. Consider the function $\psi_t(x) = (1 - \phi(x))e^{-tx}$. Then $\psi_t \overset{t\to\infty}{\to} 0$, so by functional calculus theory we have that $\psi_t(D^2 \mid_{S^i}) = (1 - \phi(D^2 \mid_{S^i})e^{-tD^2|_{S^i}} = e^{-tD^2|_{S^i}} - P^i \overset{t\to\infty}{\to} 0$. The claim about the smoothing operators follows since the map from a functional to its smoothing operator is continuous. □

Naively, we then have the result that $\operatorname{tr}(FP^i) = \lim_{t\to\infty} \operatorname{tr}(Fe^{-tD^2|_{S^i}})$, hence

$$L(f,\zeta) = \lim_{t\to\infty} \sum_i (-1)^i \operatorname{tr}(Fe^{-tD^2|_{S^i}}). \tag{5.1}$$

But this requires some justification. In fact, the key lies in considering instead the smoothing kernels.

Claim 5.2. $\operatorname{tr}(FP^i) = \lim_{t\to\infty} \operatorname{tr}(Fe^{-tD^2|_{S^i}})$.

Proof. Let $k_t(m_1, m_2)$ denote the smoothing kernel of $e^{-tD^2|_{S^i}}$. Then, since this functional is smoothing, it is of trace-class. Hence

$$\operatorname{tr}(Fe^{-tD^2|_{S^i}}) = \int \operatorname{tr} Fk_t(m,m)\operatorname{vol}(m).$$

The limit as $t \to \infty$ passes inside of the integral since M is compact and the integrand converges uniformly. This completes the proof. □

So now we wish to work with equation 5.1. Happily we can also discard the limit since the expression in the sum is independent of t for $t > 0$. The argument goes as follows:

Claim 5.3. $\frac{d}{dt}\sum_i (-1)^i \operatorname{tr}(Fe^{-tD^2|_{S^i}}) = 0$.

Proof. By an argument similar to the previous claim, we can naively perform the derivative to produce the expression

$$\sum_i (-1)^i \operatorname{tr}(FD^2 e^{-tD^2|_{S^i}}) = \sum_i (-1)^i \operatorname{tr}(F(dd* + d*d)e^{-tD^2|_{S^i}}).$$

By assumption $dF = Fd$, hence

$$\begin{array}{ccc}
\operatorname{tr}(Fdd*e^{-tD^2|_{S^i}}) & = & \operatorname{tr}(dFd*e^{-tD^2|_{S^i}}) \\
 & = & \operatorname{tr}(dFd*e^{-tD^2/2|_{S^i}}e^{-tD^2/2|_{S^i}}) \\
\text{cyclicity} & = & \operatorname{tr}(e^{-tD^2/2|_{S^i}}dFd*e^{-tD^2/2|_{S^i}}) \\
\text{cyclicity} & = & \operatorname{tr}(Fd*e^{-tD^2/2|_{S^i}}e^{-tD^2/2|_{S^i}}d) \\
\text{cyclicity} & = & \operatorname{tr}(Fd*e^{-tD^2|_{S^i}}d).
\end{array}$$

But $\operatorname{tr}(Fd*e^{-tD^2|_{S^i}}d) = \operatorname{tr}(Fd*de^{-tD^2|_{S^{i-1}}})$, hence the terms in the sum above cancel in pairs, completing the proof. □

Compiling these results, we now have the formula

$$L(f,\zeta) = \sum_i (-1)^i \operatorname{tr}(Fe^{-tD^2|_{S^i}}). \tag{5.2}$$

5.1. Contribution from points moved by $f : M \to M$.

In this section, we will show that the non-fixed points of f contribute nothing to equation 5.2, thereby inching us closer to our goal of finding the Lefschetz number in terms of the fixed points.

The key is to study this equation for small $t > 0$. Again let $k_t(m_1, m_2)$ denote the smoothing kernel of $e^{-tD^2}|_{S^i}$. Then k_t is a section of the tensor product bundle $S \boxtimes S^* \equiv \pi_1^* S \otimes \pi_2^* S^*$ where π_1 and π_2 denote the canonical projections from $M \times M$ to M.

It is not difficult to see that the smoothing kernel of $Fe^{-tD^2}|_{S^i}$ is then just $\zeta \mid_{S^i} k_t(f(m_1), m_2)$ where ζ acts on the first "coordinate" of $S \boxtimes S^*$. Denote $\zeta_i(p) \equiv \zeta(p) \mid_{S^i}$.

Then

$$\text{tr}\,(Fe^{-tD^2}|_{S^i}) = \int \text{tr}\,(\zeta_i k_t(f(m), m))\text{vol}\,(m), \tag{5.3}$$

and since we are now assuming that $m \in M$ is *not* a fixed point of f, $k_t(f(m), m) \overset{t\to 0}{\to} 0$ uniformly in m. This follows from, for instance, the asymptotic expansion for the smoothing kernel which shows that the kernel vanishes uniformly off of the diagonal as $t \to 0$ [1].

5.2. Contribution from fixed points.

Recall that we are limiting ourselves to the case that the fixed points p are simple, i.e. $\det(1 - T_p f) \neq 0$. First, we will need the following result for Gaussian integrals, which is easily verified by the reader (using, for instance, spectral decomposition):

Claim 5.4. Let T be an $n \times n$ matrix. Then

$$\frac{1}{(4\pi t)^{n/2}} \int e^{-|x - Tx|^2/4t} d^n x = \frac{1}{\mid \det(1 - T) \mid}$$

for any $t > 0$.

Now we are ready to prove the main formula in equation 2.3. The best place to start is equation 5.3, repeated again for convenience,

$$\text{tr}\,(Fe^{-tD^2}|_{S^i}) = \int \text{tr}\,(\zeta_i k_t(f(m), m))\text{vol}\,(m).$$

By the argument in the last subsection, we only need to consider this trace in a small neighborhood surrounding the fixed points p.

So pick a fixed point p, and pick a geodesic coordinate system x^i around this point. Denote $T_0 f = T$ and $g = \det g_{ij}$. Then it is easy to compute that the following estimates hold:

$$\begin{array}{rcl} \zeta_i(x) & = & \zeta_i(0) + O(\mid x \mid) \\ f(x) & = & Tx + O(\mid x \mid^2) \\ g(x) & = & 1 + O(\mid x \mid). \end{array} \tag{5.4}$$

Furthermore, the first-order terms in the asymptotic expansion for $k_t(m_1, m_2)$ around p are (see [1])

$$k_t(m_1, m_2) \sim \frac{1}{(4\pi t)^{n/2}} \exp(\frac{- \mid m_1 - m_2 \mid^2}{4t})(\Theta_0(m_1, m_2) + O(t)) + O(t). \tag{5.5}$$

where $\Theta_0(m, m) = 1$.

Substituting equation 5.4 into equation 5.5 yields the following estimate:

$$k_t(f(x),x) \sim \frac{1}{(4\pi t)^{n/2}} \exp(\frac{-\mid x - Tx \mid^2}{4t})(1 + O(\mid x \mid) + O(t) + O(\mid x \mid^3 /t)) + O(t).$$

Since we limited ourselves to simple fixed points, we have $\det(1-T) \neq 0$, hence there exists a constant $\delta > 0$ such that $\mid x - Tx \mid^2 \geq \delta \mid x \mid^2$.

Using this we obtain the estimate

$$\begin{aligned} & \mid \zeta_i k_t(f(x),x)\sqrt{g}(x) - \tfrac{\zeta_i(0)}{(4\pi t)^{n/2}} \exp(\tfrac{-|x-Tx|^2}{4t}) \mid \\ \leq \; & \tfrac{1}{(4\pi t)^{n/2}} \exp(\tfrac{-\delta |x|^2}{4t})(O(\mid x \mid) + O(t) + O(\mid x \mid^3 /t)) + O(t). \end{aligned}$$

A small calculation will verify that the L^1 norm of the RHS is of order $t^{1/2}$, and hence the RHS vanishes in the L^1 norm as $t \to 0$. Hence, in the limit we obtain

$$\int \operatorname{tr}(\zeta_i k_t(f(x),x))\sqrt{g}(x) d^n x \to \int \operatorname{tr}(\frac{\zeta_i(0)}{(4\pi t)^{n/2}} \exp(\frac{-\mid x - Tx \mid^2}{4t})) d^n x.$$

Moving the integral inside of the trace and applying the Gaussian formula 5.4 we see that

$$\text{RHS} = \frac{(-1)^i \operatorname{tr}(\zeta(0) \mid_{S^i})}{\mid \det(1-T) \mid}.$$

Adding up the contribution from all of the fixed points yields the formula

$$\operatorname{tr}(Fe^{-tD^2}|_{S^i}) \to \sum_p \frac{(-1)^i \operatorname{tr}(\zeta(p) \mid_{S^i})}{\mid \det(1 - T_p f) \mid},$$

which is just equation 2.3 after summing over i.

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