A BRIEF GUIDE TO ORDINARY K-THEORY

SPENCER STIRLING

ABSTRACT. In this paper we describe some basic notions behind ordinary K-theory. Following closely [Hat02b] we'll first study unreduced K-theory. Without working too hard we'll see that unreduced K-theory exhibits cohomological properties; in particular we shall produce an *external* product map analogous to the cross product in ordinary cohomology. The remarkable fact, known as Bott periodicity, is that this map is actually an isomorphism in certain important cases. We'll exploit these important cases as we move on and study reduced K-theory. Like unreduced K-theory, reduced K-theory exhibits cohomological properties, and Bott periodicity will show us that the usual cohomological long exact sequences (in the case of K-theory) are actually periodic.

1. Introduction

In this paper we hope to provide a brief introduction to the ideas behind ordinary K-theory. Since the goal of this paper is to provide a whirlwind tour we shall omit many details. In particular we assume that the reader is already familiar with the basic vector bundle constructions such as direct sum and tensor product of vector bundles (see Chapter 1 from [Hat02b] for details).

Also in the interest of clarity we shall not sprinkle repeated references to [Hat02b] throughout this paper. Anything not rigorous here is presented there in beautiful detail (in fact we'll keep identical notation).

2. Unreduced K-theory

To begin fix a compact Hausdorff manifold X. It's useful to set some notation before charging ahead: denote the trivial n-dimensional complex vector bundle over X as $\varepsilon^n \in \mathrm{Vect}_{\mathbb{C}}(X)$ (from now on we understand that all vector bundles in this paper are *complex*).

Our goal is to construct a *ring* out of the operations of direct sum $\oplus$ and tensor product $\otimes$ on $\mathrm{Vect}_{\mathbb{C}}(X)$ (despite the fact that there is no ring structure from the outset).

Given an arbitrary vector bundle V it is trivial to see that $V \oplus \varepsilon^0 \simeq \varepsilon^0 \oplus V \simeq V$ and also that $V \otimes \varepsilon^1 \simeq \varepsilon^1 \otimes V \simeq V$. So we have an abelian monoidal operation $\oplus$ and a commutative multiplication operation $\otimes$ (with unit). In other words we have a structure that almost looks like a commutative ring with identity.

Since we're very close to having a ring structure it makes sense to try to extend the abelian monoidal structure to an abelian group structure. It turns out that this can always be done, at least formally, using the standard Grothendieck construction.

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The structure that is missing from what we already have is an additive inverse (i.e. *negative* vector bundles). The consistent way to extend the category is to consider *formal differences* of vector bundles $E_1 - E_2 \in \mathrm{Vect}_{\mathbb{C}}(X) \times \mathrm{Vect}_{\mathbb{C}}(X)$.

Obviously the space $\mathrm{Vect}_{\mathbb{C}}(X) \times \mathrm{Vect}_{\mathbb{C}}(X)$ is far too big for our purposes because (as the notation suggests) we *want* to identify two formal differences $E_1 - E_2 = E_1' - E_2'$ if

$$E_1 \oplus E_2' \simeq E_1' \oplus E_2, \tag{1}$$

but even this space it too big! It turns out that demanding a vector bundle isomorphism $\simeq$ in the equation above is far too discerning for two ordered pairs to be "equivalent". We can, however, weaken this condition considerably. For this let us introduce the (more forgiving) notion of *stable isomorphism*: write $V \approx W$ if there exists a trivial bundle ε^n for some n such that $V \oplus \varepsilon^n \simeq W \oplus \varepsilon^n$.

Now we have the appropriate ideas in place. We'll set two ordered pairs equal $E_1 - E_2 = E_1' - E_2'$ if (note the *stable* isomorphism - we do not require an actual isomorphism!)

$$E_1 \oplus E_2' \approx E_1' \oplus E_2. \tag{2}$$

The set of formal differences has a well-defined addition:

$$(E_1 - E_2) + (E_1' - E_2') = E_1 \oplus E_1' - E_2 \oplus E_2' \tag{3}$$

a well-defined negation:

$$-(E_1 - E_2) = E_2 - E_1 \tag{4}$$

and an additive identity

$$E - E. \tag{5}$$

(it's easy to check that this equivalence class is independent of E).

Multiplication is defined by using the obvious distributive law

$$(E_1 - E_2)(E_1' - E_2') = ((E_1 \otimes E_1') \oplus (E_2 \otimes E_2')) - ((E_1 \otimes E_2') \oplus (E_2 \otimes E_1')) \tag{6}$$

and the multiplicative identity is just (the equivalence class of)

$$\varepsilon^1 - \varepsilon^0. \tag{7}$$

We denote this *ring* of formal differences $K(X)$, the unreduced K-theory of X.

3. Some similarities to ordinary cohomology

Recall that ordinary cohomology [Hat02a] is a contravariant functor from the category of topological spaces to the category of rings. More concretely if $f : X \to Y$ is a continuous map between topological spaces then there is an induced ring homomorphism $f^* : H^*(Y) \to H^*(X)$ (satisfying the usual properties). These pullbacks are only sensitive up to homotopy classes of maps, i.e if f is homotopic to g ($f \simeq g$) then $f^* = g^*$.

In ordinary cohomology we first meet the *cup product* $\smile$ which tells us how to multiply two elements in the cohomology over the *same* space X. The analogue in K-theory is the (interior) product described in Equation 6.

Cohomology, however, is much richer and admits several types of products. For example, there is the so-called *cross product* which relates the cohomology of a product space $H^*(X \times Y)$ to the cohomology of the individual factors $H^*(X)$ and $H^*(Y)$. It's not surprising that we define this using the cup product

$$\times : H^*(X) \times H^*(Y) \to H^*(X \times Y) \tag{8}$$
$$(a, b) \mapsto p_1^*(a) \smile p_2^*(b) \tag{9}$$

where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are the usual projection maps.

Furthermore it's easy to see that the cross product is actually R-linear (where R is the coefficient ring) since $\smile$ distributes over addition. This means that the cross product factors through the following diagram

$$\begin{array}{ccc} H^*(X) \times H^*(Y) & \longrightarrow & H^*(X \times Y) \\ & \searrow \quad \nearrow & \\ & H^*(X) \otimes_R H^*(Y) & \end{array} \tag{10}$$

We therefore write the cross product as a map

$$\times : H^*(X) \otimes_R H^*(Y) \to H^*(X \times Y) \tag{11}$$
$$a \otimes_R b \mapsto p_1^*(a) \smile p_2^*(b) \tag{12}$$

As we shall now see it happens that unreduced K-theory admits natural analogues of the structures just discussed. This encourages us to view K-theory as an example of a *generalized* cohomology theory (in the sense of Eilenberg-Steenrod). Reduced K-theory, which we'll meet in the sequel, actually mimics ordinary cohomology even better!

K-theory can, like ordinary cohomology, be viewed as a contravariant functor from the category of compact Hausdorff topological manifolds to the category of rings. In other words, a continuous map $f : X \to Y$ induces a pullback map

$$f^* : K(Y) \to K(X) \tag{13}$$
$$E - E' \mapsto f^*(E) - f^*(E'). \tag{14}$$

It is easy to check that the pullback map f^* is a ring homomorphism, and it also follows from the properties of vector bundles that $(fg)^* = g^* f^*$ and $\mathbb{1}^* = \mathbb{1}$. Furthermore, if f is homotopic to g ($f \simeq g$) then $f^* = g^*$. So K-theory indeed defines a contravariant functor from (compact Hausdorff) topological manifolds to rings.

Just as we have a cross product in ordinary cohomology we see that an *external* product emerges in unreduced K-theory

$$\mu : K(X) \times K(Y) \to K(X \times Y) \tag{15}$$
$$(a, b) \mapsto p_1^*(a) p_2^*(b) \tag{16}$$
$$\tag{17}$$

where on the RHS we have used the (internal) product previously defined in Equation 6 and $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$ are the usual projection maps. In fact (much like the cross product in ordinary cohomology) it is straightforward to check that multiplication factors through the diagram

$$K(X) \times K(Y) \longrightarrow K(X \times Y), \qquad K(X) \times K(Y) \to K(X) \otimes K(Y) \to K(X \times Y) \tag{18}$$

so we view (external) multiplication as a map

$$\mu : K(X) \otimes K(Y) \to K(X \times Y) \tag{19}$$

$$a \otimes b \mapsto p_1^*(a) p_2^*(b) \tag{20}$$

$$\tag{21}$$

For brevity let's agree to use the notation

$$\mu(a \otimes b) = a * b. \tag{22}$$

4. Bott periodicity

The fundamental insight that allows us to proceed further is the following beautiful result discovered by Bott [Bot59]:

Theorem 1 (Bott)**.** *The external product $\mu : K(X) \otimes K(S^2) \to K(X \times S^2)$ is a ring isomorphism.*

Proof. Omitted. See [Hat02b]. □

Remark 1. Notice that the second factor in the theorem is $Y = S^2$. Naively this result seems "cute" but perhaps not generally useful. In fact this result will mean a great deal more later.

5. Reduced K-theory

Now we transition to the second theme of this paper: reduced K-theory $\tilde{K}(X)$. For the purposes of this paper there are two vantagepoints on $\tilde{K}(X)$. The first is as follows: recall that in unreduced K-theory we needed to talk about (equivalence classes of) formal differences of vector bundles. This was necessary because we didn't have additive inverses. We intimated that our notion of "equivalence" of formal differences required the notion of *stable* isomorphism $\approx$ (recall that two vector bundles $V, W \in \mathrm{Vect}_{\mathbb{C}} X$ are stably isomorphic $V \approx W$ if there exists a trivial vector bundle ε^n such that $V \oplus \varepsilon^n \simeq W \oplus \varepsilon^n$ are isomorphic).

As far as reduced K-theory is concerned our tak is somewhat similar. Here we will *not*, however, need to talk about formal differences of vector bundles. Rather we can proceed with $\mathrm{Vect}_{\mathbb{C}} X$ itself. The cost of doing so is that our notion of equivalence will need to be *even weaker* than stable isomorphism.

For $V, W \in \mathrm{Vect}_{\mathbb{C}} X$ define $V \sim W$ if there exists trivial bundles ε^n and ε^m (of possibly different dimension) such that $V \oplus \varepsilon^n \simeq W \oplus \varepsilon^m$. Denote the set of equivalence classes by $\tilde{K}(X)$. With respect to $\oplus$ the additive identity is ε^m for arbitrary m since $\varepsilon^0 \sim \varepsilon^m$. In view of the following lemma $\tilde{K}(X)$ has additive inverses "for free", so $\tilde{K}(X)$ is already an abelian group (without having to appeal to the Grothendieck construction):

Lemma 1. *For every vector bundle E over a compact Hausdorff space X there exists a vector bundle E' such that $E \oplus E'$ is trivial.*

Proof. Omitted. See [Hat02b]. □

In fact $\tilde{K}(X)$ happens to be a ring (even though we haven't written down a well-defined multiplication operation yet).

To see that $\tilde{K}(X)$ is a ring we need our second vantagepoint on reduced K-theory. Let us return momentarily to unreduced K-theory $K(X)$. Using Lemma 1 it is easy to show that any element of $K(X)$ has a representative of the form $E - \varepsilon^n$ (where as usual ε^n is a trivial vector bundle).

Then we can define a group homomorphism $K(X) \to \tilde{K}(X)$ as the map $E - \varepsilon^n \mapsto E$. It is trivial to show that this map is independent of choice of representative since if $E - \varepsilon^n \approx E' - \varepsilon^m$ then $E \sim E'$. The kernel of this homomorphism is the subgroup $\{\varepsilon^m - \varepsilon^n\} \simeq \mathbb{Z}$.

Stated more formally, reduced and unreduced K-theory fit into the following short exact sequence

$$0 \to \mathbb{Z} \underset{p}{\overset{s}{\rightleftarrows}} K(X) \to \tilde{K}(X) \to 0. \tag{23}$$

Our notation alludes to something quite stronger: there exists a splitting map p; in other words $\tilde{K}(X) \oplus \mathbb{Z} \simeq K(X)$.

Let us construct the splitting map p. Before we begin choose a basepoint $x_0 \in X$. We easily convince ourselves that $K(x_0) \simeq \mathbb{Z}$ since every vector bundle over a single point is trivial. Therefore we substitute $K(x_0)$ for $\mathbb{Z}$ in the short exact sequence above.

Now the inclusion map $x_0 \hookrightarrow X$ induces a pullback map $p : K(X) \to K(x_0)$. On the other hand the map $X \to x_0$ that crushes X to a point induces its own pullback map $s : K(x_0) \to K(X)$. It is straightforward to check that $p \circ s$ is the identity, so we have succeeded in constructing the desired splitting $\tilde{K}(X) \oplus K(x_0) \simeq K(X)$. In view of this we have our second characterization $\tilde{K}(X) \simeq \frac{K(X)}{K(x_0)}$.

Looking more closely, it's easy to see that $K(x_0)$ is an ideal in $K(X)$, hence the above quotient actually makes sense as a ring quotient. This is how we equip $\tilde{K}(X)$ with a well-defined multiplication. So $\tilde{K}(X)$ is a ring.

6. Some similarities to ordinary cohomology

Using the second characterization of $\tilde{K}(X)$ it seems that reduced K-theory is more naturally associated with pointed spaces. In this light reduced K-theory plays a role analogous to what reduced cohomology plays when considering ordinary cohomology theories.

For example in Section 3 we produced an external product map $\mu : K(x) \otimes K(Y) \to K(X \times Y)$. A similar external product β arises in reduced K-theory, except for pointed spaces the natural analogue to $X \times Y$ is the *smash product* $X \wedge Y \equiv X \times Y / X \vee Y$.

Remark 2. $X \vee Y$ is the usual wedge product gluing two pointed spaces together at their basepoints: $(X \times \{y_0\}) \cup (Y \times \{x_0\})$.

It's easy to describe this external product: let $a \in \tilde{K}(X) = \frac{K(X)}{K(x_0)}$ and $b \in \tilde{K}(Y) = \frac{K(Y)}{K(y_0)}$. Then it is a matter of definition chasing to show that, under pullback of the projection map $p_1 : X \times Y \to X$ we have $p_1^*(a) \in \frac{K(X \times Y)}{K(x_0 \times Y)}$. Similarly $p_2^*(b) \in \frac{K(X \times Y)}{K(y_0 \times X)}$.

Since $K(x_0 \times Y)$ and $K(y_0 \times X)$ are ideals we then see that

$$p_1^*(a)p_2^*(b) \in \frac{K(X \times Y)}{K(x_0 \times Y) + K(y_0 \times X)}. \tag{24}$$

Now $\{x_0 \times Y\}$ and $\{y_0 \times X\}$ intersect at only a single point so we see that $K(x_0 \times Y) + K(y_0 \times X) = K(\{x_0 \times Y\} \cup \{y_0 \times X\}) = K(X \vee Y)$. We are left with

$$p_1^*(a)p_2^*(b) \in \frac{K(X \times Y)}{K(X \vee Y)} = \frac{\frac{K(X \times Y)}{K(x_0 \times y_0)}}{\frac{K(X \vee Y)}{K(x_0 \times y_0)}} = \frac{\tilde{K}(X \times Y)}{\tilde{K}(X \vee Y)} = \tilde{K}(X \wedge Y). \tag{25}$$

The reader will probably riot (rightfully so) at the last equality. A short glance below to Fact 1 should give us some direction to resolve the confusion. We get an exact sequence $\tilde{K}(X \wedge Y) \to \tilde{K}(X \times Y) \to \tilde{K}(X \vee Y)$. The magic is that this sequence actually splits, which gives us the last equality above. This splitting is left to the reader to construct (hint: use the tools given in the next section to show that $\tilde{K}(X \vee Y) \simeq \tilde{K}(X) \oplus \tilde{K}(Y)$)

We have now successfully defined the reduced version of the exterior product.

7. Long exact sequence

As usual we want to consider long exact sequences for our "cohomology" theory. Like all homological algebra stories, this one requires a great deal of setup. We start by collecting some relevant facts.

Fact 1. *If A is a closed subspace of a compact Hausdorff space X then the inclusion and quotient maps $A \xrightarrow{i} X \xrightarrow{q} X/A$ induce an exact sequence (via pullback) $\tilde{K}(X/A) \to \tilde{K}(X) \to \tilde{K}(A)$.*

Already this is starting to look like cohomology. Perhaps we're losing something by not producing the proof of the above fact here. Rest assured that the proof does not reflect a mysterious fact about reduced K-theory, but rather it's a "nuts-and-bolts" construction on the underlying vector bundles in reduced K-theory. At least the following fact is exceedingly easy to believe:

Fact 2. *If A is contractible then the quotient map $X \to X/A$ induces a bijection $Vect^n_{\mathbb{C}}(X/A) \to Vect^n_{\mathbb{C}}(X)$.*

The next two facts can be used to complete the exercise left to the reader at the end of the last section. They can be skipped on a first reading.

Fact 3. $\tilde{K}(U \wedge V) \simeq \tilde{K}(U) \oplus \tilde{K}(V)$

Proof. Let $X = U \wedge V$ and $A = U$. Fact 1 becomes in this form the short exact sequence

$$\tilde{K}(U) \to \tilde{K}(U \wedge V) \to \tilde{K}(V). \tag{26}$$

It is a simple geometrical fact that this sequence splits since U and V are connected only at a single point (where vector bundles are trivial). In other words the equivalence class $\sim$ of elements in $\mathrm{Vect}_{\mathbb{C}}U$ has basically nothing to do with the equivalence class of elements in $\mathrm{Vect}_{\mathbb{C}}V$ in the wedge product $U \wedge V$ because they're glued together where vector bundles are trivial. □

This and Fact 2 imply a similar result about suspensions:

Fact 4. $\tilde{K}(S(U \wedge V)) \simeq \tilde{K}(S(U)) \oplus \tilde{K}(S(V))$

Proof. It's trivial to see that $S(U \wedge V) \simeq S(U) \wedge S(V)$ (U and V are connected at a single point. Taking the suspension means crossing with an interval, which is contractible). □

Armed with these facts, let us proceed and try to produce some long exact sequences. First, look at Fact 1. In ordinary cohomology theories we are accustomed to converting (chains of) such short exact sequences into long exact sequences, so we may expect a similar kind of homological algebraic construction here.

Although we have short exact sequences, unfortunately we do not have (yet) any species of long exact sequence naturally emerging. Fortunately we can remedy this situation rather easily. Consider the following map of inclusions known as the Puppe sequence:

$$A \hookrightarrow X \hookrightarrow X \cup CA \hookrightarrow (X \cup CA) \cup CX \hookrightarrow ((X \cup CA) \cup CX) \cup C(X \cup CA) \hookrightarrow \cdots \tag{27}$$

where C denotes the usual "cone" operation (crossing a space with the unit interval and crushing one end to a point). Each space is constructed by taking the space directly to the left and coning off the (already included) subspace that sits two slots to the left.

A few mental pictures will convince us of the following homotopies (just use the cone to crush the subspace included from two slots to the left to a point):

$$X \cup CA \simeq X/A \tag{28}$$

$$(X \cup CA) \cup CX \simeq SA \tag{29}$$

$$((X \cup CA) \cup CX) \cup C(X \cup CA) \simeq SX. \tag{30}$$

(31)

S denotes the suspension of a space, that is crossing a space with the unit interval and crushing the ends (each separately) to a point. In light of this homotopy the above inclusions take the form

$$A \hookrightarrow X \hookrightarrow X/A \hookrightarrow SA \hookrightarrow SX \hookrightarrow \cdots \tag{32}$$

Obviously the inclusions written here are not *actual* inclusions, but rather are homotopic to inclusions.

Nevertheless this induces a long exact sequence

$$\cdots \to \tilde{K}(S^2X) \to \tilde{K}(S^2A) \to \tilde{K}(S(X/A)) \to \tilde{K}(SX) \to \tilde{K}(SA) \to \tilde{K}(X/A) \to \tilde{K}(X) \to \tilde{K}(A) \tag{33}$$

where S^2X denotes the suspension of the suspension of X (and so on).

A slight change of notation is in order, especially to bring our results more in line with ordinary cohomology. Ordinary cohomology is $\mathbb{Z}$-graded, and the usual long exact sequences respect this grading, that is H^0 maps to H^1, H^1 maps to H^2, and so on. We can achieve a similar notation if we denote $\tilde{K}^{-n}(X) \equiv \tilde{K}(S^nX)$. Then the above long exact sequence becomes

$$\cdots \to \tilde{K}^{-2}(X) \to \tilde{K}^{-2}(A) \to \tilde{K}^{-1}(X/A) \to \tilde{K}^{-1}(X) \to \tilde{K}^{-1}(A) \to \tilde{K}^{0}(X/A) \to \tilde{K}^{0}(X) \to \tilde{K}^{0}(A) \tag{34}$$

Remark 3. We use negative powers to fit with the usual notation that the grading should *increase* as one travels up the long exact sequence.

Table 1. Stable Homotopy Groups

$\pi_{n-1}\mathrm{U}(k)$

		$k \rightarrow$				
		1	2	3	4	...
$n-1$	0	0	0	0	0	...
$\downarrow$	1	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	...
	2	0	0	0	0	...
	3	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	...

We'll switch freely between this "graded" notation and the previous "n-fold suspension" notation.

8. K-theory of spheres

We have managed to shamefully pass through a large chunk of formalism without giving any examples. In this section we'll calculate the K-theory for the spheres using a plethora of facts. Hopefully the reader will get the correct impression that K-theory can be difficult to calculate in general. We also remark that the sphere example S^2 is actually used in the proof of Bott periodicity. Since we haven't actually *used* Bott periodicity for anything yet this is not circular reasoning.

In order to calculate K-theory for the spheres it makes sense to attempt to first describe the vector bundles over the spheres. Consider the following: the sphere S^n can be constructed by gluing two disks D^n_+ and D^n_- along the equator S^{n-1}. Since the disks are contractible any vector bundle over each disk must be trivial. Therefore the vector bundles over S^n are completely described by the *clutching functions* which tell us how to glue the vector bundles together along the equator S^{n-1}.

In other words to glue two k-dimensional vector bundles over the disks together we need to give at each point along the equator S^{n-1} an element of $\mathrm{GL}_{\mathbb{C}}(k)$ (i.e. we need to specify a map $S^{n-1} \rightarrow \mathrm{GL}_{\mathbb{C}}(k)$). On the other hand it is a fact that the resulting vector bundles are only determined by such a map *up to homotopy*, hence actually the k-dimensional vector bundles $\mathrm{Vect}^k_{\mathbb{C}}(S^n)$ are classified by $\pi_{n-1}(\mathrm{GL}_{\mathbb{C}}(k))$.

We can further reduce the description if we recall that $\mathrm{GL}_{\mathbb{C}}(k)$ deformation retracts onto $\mathrm{U}(k)$. This is easily seen from the following: it is certainly true that the Gram-Schmidt process retracts $\mathrm{GL}_{\mathbb{C}}(k)$ onto $\mathrm{U}(k)$. The reader can easily cook up a scheme using scalar parameters to convert this retraction into a deformation retraction.

So we now have a bijection $\pi_{n-1}(\mathrm{U}(k)) \longleftrightarrow \mathrm{Vect}^k_{\mathbb{C}}(S^n)$ (the fact that this is a bijection is left to the references). On the other hand $K(S^n)$ and $\tilde{K}(S^n)$ hold information about $\mathrm{Vect}^k_{\mathbb{C}}(S^n)$ for *every* dimension k, so we might want to think about somehow talking about all dimensions at once!

Consider the space $\mathrm{U} \equiv \cup_k \mathrm{U}(k)$. Here each $\mathrm{U}(k)$ is embedded in $\mathrm{U}(k+1)$ by adjoing to each k-dimensional matrix an extra row and column with a single 1 in the corner. The concerned reader should give U the weak topology.

Technicalities aside, our dreams come true:

Fact 5. *$\tilde{K}(S^n)$ is isomorphic (as a group) to $\pi_{n-1}(U)$*

Our good fortunate doesn't end there! We can actually compute $\pi_{n-1}(\mathrm{U})$ if were not too ambitious (i.e. if $n-1$ is small enough - and we'll relax even *that* at the end of the paper). Look at Table 1. Note that the homotopy groups don't appear to change as we increase the dimension k. This is known as *stability*.

Although homotopy groups can be hard to compute in general, it is true that homotopy groups have long exact sequences (much like homology and cohomology). In the case where our space is $\mathrm{U}(k)$ much can be said from looking at the long exact sequences of homotopy groups. Incredibly the apparent stability of the groups $\pi_{n-1}(\mathrm{U}(k))$ is actually correct for k big enough:

Fact 6. *The map $\pi_{n-1}(U(k)) \to \pi_{n-1}(U(k+1))$ induced by the inclusion $U(k) \hookrightarrow U(k+1)$ is an isomorphism if $k > \frac{n-1}{2}$.*

If we "pass through the limit" as $k \to \infty$ then the stability tells us what $\pi_{n-1}(\mathrm{U})$ must be.

Let's be more concrete for a moment. Consider the space S^2. We know now that $\tilde{K}(S^2) \simeq \pi_1(\mathrm{U})$, and from Table 1 we see that $\pi_1(\mathrm{U}) \simeq \mathbb{Z}$.

9. Bott periodicity: revisited

Recall that we already produced an external product

$$\beta : \tilde{K}(X) \oplus \tilde{K}(Y) \to \tilde{K}(X \wedge Y) \tag{35}$$

before we became sidetracked with long exact sequences and the sphere examples.

At this point it makes sense to try to come up with a version of Bott periodicity for this version of external multiplication. The direct analogue is

Theorem 2 (Bott). *$\beta : \tilde{K}(X) \otimes \tilde{K}(S^2) \to \tilde{K}(S^2 \wedge X)$ is a ring isomorphism.*

From our previous discussion we know that $\tilde{K}(S^2) \simeq \mathbb{Z}$, and a moment of visualization might convince us that $S^2 \wedge X$ is homotopic to $S^2 X$ (the two-fold suspension of X), so Bott periodicity becomes

Corollary 1 (Bott). *The map β induces an isomorphism $\tilde{K}(X) \simeq \tilde{K}(S^2 X) \equiv \tilde{K}^{-2}(X)$*

In light of Corollary 1 it is easy to see that Equation 34 becomes periodic:

$$\begin{array}{ccccc} \tilde{K}^{-1}(X/A) & \longrightarrow & \tilde{K}^{-1}(X) & \longrightarrow & \tilde{K}^{-1}(A) \\ \uparrow & & & & \downarrow \\ \tilde{K}^{0}(A) & \longleftarrow & \tilde{K}^{0}(X) & \longleftarrow & \tilde{K}^{0}(X/A) \end{array} \tag{36}$$

Hence we have explained the terminology "Bott periodicity" and provided a powerful tool for actually computing K-theory.

10. Bott periodicity returns the favor to stable homotopy groups

It would be a shame to not at least mention this last result. Recall that we mentioned that the stable homotopy group for S^2 is used to *prove* Bott periodicity.

With Bott periodicity under our belt we can turn the table and see what it has to say about stable homotopy groups. Consider Corollary 1 which says that $\tilde{K}(X) \simeq \tilde{K}(S^2 X)$. In the particular case that X is a sphere we see that $\tilde{K}(S^n) \simeq \tilde{K}(S^{n+2})$.

On the other hand recall that $\tilde{K}(S^n) \simeq \pi_{n-1}(\mathrm{U})$. So this shows that the homotopy groups $\pi_{n-1}(\mathrm{U})$ repeat every other n.

This is extremely surprising: not only does $\pi_{n-1}(\mathrm{U}(k))$ stabilize as $k \to \infty$, but it repeats every other n as well (for k large enough)!

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Department of Mathematics, University of Texas at Austin