RATIOS OF FUNCTIONAL DETERMINANTS USING GREEN FUNCTIONS

SPENCER STIRLING

ABSTRACT. We follow the method of H. Kleinert and A. Chervyakov to calculate the ratio of functional determinants of second-order differential operators. Their method relies on the Wronski formulation for Green functions. In particular we present a formulation for Dirichlet, periodic, and antiperiodic boundary conditions.

1. INTRODUCTION

It is often the case, especially in quantum field theory (but also in condensed matter physics and stochastic processes), that we are required to evaluate a Gaussian path integral (see, for example, [6]), i.e. we want to calculate

$$\int Dx \exp\left(\int dt \frac{1}{2}(x, Kx)\right) \tag{1.1}$$

where K is some second-order differential operator, and x is some function of t (say, position). $\int Dx$ denotes a formal expression which can be interpreted as an integral over paths. It is approximated by discretizing time (i.e. $t \to t_i$), and in fact it is *defined* as the continuum limit of the discrete problem.

Then, in this approximation, we have $Dx \to \Pi dx_i$ where x_i denotes $x(t_i)$, and the integral in the exponent turns to a sum. In this case K is just some matrix, and the path integral becomes just a standard integral over multiple dimensions. This integral is well known, and is just

$$[\det K]^{-1/2}.$$

So by analogy, we *define* the value of the path integral (if K is second-order, i.e. Gaussian) to be given by this determinant, where now K is some differential operator.

Thus, in general, we require a method of finding the determinant for differential operators, i.e. the *functional* determinant. Much work has been devoted to this, most notably in the general solutions given by Gel'fand and Yaglom [5]. Unfortunately these methods can be complicated.

Fortunately, however, we usually want only a *normalized* path integral, i.e. we want to divide any path integral like in equation (1.1) by some fixed Gaussian path integral. Hence, what we *really* want is the ratio of functional determinants.

In this paper we explore a simple method proposed by H. Kleinert and A. Chervyakov [2] to calculate such ratios using the Wronski formulation for Green functions. We present explicit solutions for the case of Dirichlet, periodic, and

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Date: December 7, 2003.

antiperiodic boundary conditions, and we present the formulas for the harmonic oscillator.

2. Ratio Formulation

Usually we find ourselves dealing with an operator of the form

$$K_1 = -\partial_t^2 - \Omega^2(t) \tag{2.1}$$

that arises in the semiclassical approximation. In this form (neglecting any mass coefficients) the derivative term corresponds to the kinetic energy part of the operator and $\Omega^2(t) = V''$ is the quadratic fluctuation of the potential evaluated around the classical path.

So the solution to our path integral is just a ratio of functional determinants, i.e.

$$\left(\frac{\det K_1}{\det \widetilde{K}}\right)^{-1/2}$$

where $\widetilde{K}$ is some fixed operator (as we will see later, in the Dirichlet case we will choose $\widetilde{K} = -\partial_t^2$, whereas in the periodic/antiperiodic cases we will choose $\widetilde{K} = -\partial_t^2 - \omega_0$ for some fixed frequency ω_0).

Using the fact that this ratio satisfies the Fredholm property we have

$$\frac{\det K_1}{\det \widetilde{K}} = \det \widetilde{K}^{-1} K_1 \tag{2.2}$$

and furthermore this determinant is well-defined [5].

We now introduce a family of operators parameterized by a variable $g \in [0,1]$ and given by

$$K_g \equiv -\partial_t^2 - g\Omega^2(t). \tag{2.3}$$

In accordance with our above notation, K_1 is our original operator. Then, using the well-known formula for the determinant in terms of the trace

$$\log\det \widetilde{K}^{-1} K_g = \operatorname{Tr}\log \widetilde{K}^{-1} K_g$$

and taking a derivative with respect to g yields

$$\partial_g \log\det \widetilde{K}^{-1} K_g = \operatorname{Tr} \frac{\widetilde{K}^{-1}}{\widetilde{K}^{-1} K_g}(-\Omega^2(t)) = -\operatorname{Tr} \Omega^2(t) G_g(t,\tau) \tag{2.4}$$

where the Green function is given by

$$K_g G_g(t,\tau) = \delta(t-\tau). \tag{2.5}$$

Integrating, we obtain

$$\det \widetilde{K}^{-1} K_g = C \exp\left\{-\int_0^g dg' \operatorname{Tr} \Omega^2(t) G_g(t,\tau)\right\}$$

which becomes (after evaluating the trace),

$$\det \widetilde{K}^{-1} K_g = C \exp\left\{-\int_0^g dg' \int_{t_a}^{t_b} dt\, \Omega^2(t) G_g(t,t)\right\}. \tag{2.6}$$

This is our main formula from which we will calculate the ratio of determinants.

3. Calculating the Green Function

In order to find the Green function defined in equation (2.5) (and required in equation (2.6)) we use Wronski's formula (see also, for example, [1]).

First consider the linear differential equation

$$K_g h_g(t) = f_g(t) \tag{3.1}$$

where K_g is given in equation (2.3). Then we propose, by superposition, a solution of the form

$$h_g(t) = \int_{t_a}^{t_b} G_g(t,\tau) f_g(\tau) d\tau \tag{3.2}$$

where $G_g(t,\tau)$ is the response of the system at time t to a unit impulse applied at time τ (we call it a Green function). We interpret the homogeneous equation $K_g h_g(t) = 0$ as the free (undriven) equation of motion, and we interpret $f_g(t)$ in equation (3.1) as an external forcing function. Then, applying K_g to both sides of equation (3.2), we obtain

$$(f_g(t) =)\ K_g h_g(t) = \int_{t_a}^{t_b} (K_g G_g(t,\tau)) f_g(\tau) d\tau.$$

This explains why we define the Green function by equation (2.5).

Now the RHS of equation (2.5) vanishes if $t \neq \tau$, hence if $\eta_g(t)$ and $\xi_g(t)$ are two linearly independent solutions of the homogenous equation, then we have

$$\begin{array}{rclcll} G_g(t,\tau) & = & a_g(\tau)\eta_g(t) + b_g(\tau)\xi_g(t) & \equiv & G_g^+(t,\tau) & t < \tau \\ G_g(t,\tau) & = & c_g(\tau)\eta_g(t) + d_g(\tau)\xi_g(t) & \equiv & G_g^-(t,\tau) & t > \tau. \end{array} \tag{3.3}$$

So we can write

$$G_g(t,\tau) = \Theta(\tau - t) G_g^+(t,\tau) + \Theta(t-\tau) G_g^-(t,\tau) \tag{3.4}$$

where $\Theta(t)$ is the Heaviside function.

Plugging this formula into equation (2.5) we obtain

$$\begin{array}{rcl} \delta(t-\tau) & = & \Theta(\tau - t) K_g G_g^+ + \left[\frac{G_g^+}{\tau - t} + 2\partial_t G_g^+\right] \delta(\tau - t) \\ & + & \Theta(t-\tau) K_g G_g^- + \left[\frac{G_g^-}{t-\tau} + 2\partial_t G_g^-\right] \delta(t-\tau) \end{array} \tag{3.5}$$

where we have used the formula $t\delta'(t) = -\delta(t)$ (see, for example, [4]).

The terms involving $\Theta(t)$ vanish since G_g^+ and G_g^- satisfy the homogeneous equation. So if we expand G_g^+ and G_g^- in the remaining terms for t around τ we obtain the conditions

$$-G_g^+ + G_g^- = 0 \tag{3.6}$$

(so G_g is continuous) and

$$\partial_t (G_g^- - G_g^+) \mid_\tau = -1 \tag{3.7}$$

(so we have a step discontinuity in the derivative at τ). These two conditions constrain the coefficient functions $a(\tau)$, $b(\tau)$, $c(\tau)$, and $d(\tau)$ found in equations (3.3).

In particular we obtain

$$\begin{aligned}(a_g(\tau)-c_g(\tau))\eta_g(\tau)+(b_g(\tau)-d_g(\tau))\xi_g(\tau) &= 0\\ (a_g(\tau)-c_g(\tau))\dot{\eta}_g(\tau)+(b_g(\tau)-d_g(\tau))\dot{\xi}_g(\tau) &= 1\end{aligned}$$

which has a solution provided the (time-independent) Wronski determinant $W_g = \eta_g\dot{\xi}_g - \dot{\eta}_g\xi_g$ is non-vanishing. This gives

$$\begin{aligned}(a_g(\tau)-c_g(\tau)) &= \tfrac{1}{W_g}(-\xi_g(\tau))\\ (b_g(\tau)-d_g(\tau)) &= \tfrac{1}{W_g}(\eta_g(\tau))\end{aligned}$$

so we obtain as a final result

$$\begin{aligned}G_g(t,\tau) &= \Theta(\tau-t)\left\{a_g(\tau)\eta_g(t)+b_g(\tau)\xi_g(t)\right\}\\ &+ \Theta(t-\tau)\left\{(a_g(\tau)+\tfrac{\xi_g(\tau)}{W_g})\eta_g(t)+(b_g(\tau)-\tfrac{\eta_g(\tau)}{W_g})\xi_g(t)\right\}.\end{aligned} \tag{3.8}$$

3.1. **Dirichlet Boundary Conditions.** If we impose Dirichlet boundary conditions on equation (3.2), i.e. $h_g(t_a) = h_g(t_b) = 0$, then we obtain two additional constraints on equation (3.8). These are

$$G_g(t_a,\tau) = 0 \Rightarrow G_g^+(t_a,\tau) = a_g(\tau)\eta_g(t_a) + b_g(\tau)\eta_g(t_a) = 0$$

and

$$G_g(t_b,\tau) = 0 \Rightarrow G_g^-(t_b,\tau) = (a_g(\tau)+\frac{\xi_g(\tau)}{W_g})\eta_g(t_b) + (b_g(\tau)-\frac{\eta_g(\tau)}{W_g})\eta_g(t_b) = 0$$

which together imply the matrix equation

$$\begin{aligned}a_g(\tau)\eta_g(t_a)+b_g(\tau)\eta_g(t_a) &= 0\\ a_g(\tau)\eta_g(t_b)+b_g(\tau)\eta_g(t_b) &= \tfrac{\eta_g(\tau)\xi_g(t_b)-\xi_g(\tau)\eta_g(t_b)}{W_g}.\end{aligned}$$

This has a solution provided

$$\det\Lambda_g = \begin{vmatrix}\eta_g(t_a) & \xi_g(t_a)\\ \eta_g(t_b) & \xi_g(t_b)\end{vmatrix} \neq 0. \tag{3.9}$$

Plugging in these new constraints yields an equation for the Green function which can be presented efficiently by introducing the function

$$\Delta_g(t,\tau) = \frac{1}{W_g}\left[\eta_g(t)\xi_g(\tau) - \xi_g(t)\eta_g(\tau)\right].$$

Then a small calculation shows that we can integrate all of our conditions into the single equation

$$G_g(t,\tau) = \frac{1}{\Delta_g(t_a,t_b)}\left\{\theta(t-\tau)\Delta_g(t_b,t)\Delta_g(\tau,t_a) + \theta(\tau-t)\Delta_g(t_b,\tau)\Delta_g(t,t_a)\right\}. \tag{3.10}$$

3.2. **Periodic and Antiperiodic Boundary Conditions.** Suppose, on the other hand, that we impose periodic or antiperiodic boundary conditions on equation (3.2), i.e.

$$h_g(t_a) = \pm h_g(t_b) \quad \dot{h}_g(t_a) = \pm\dot{h}_g(t_b).$$

This implies for the Green function the conditions

$$\begin{aligned}G_g(t_a,\tau) = \pm G_g(t_b,\tau) &\Longrightarrow G_g^+(t_a,\tau) = \pm G_g^-(t_b,\tau)\\ \dot{G}_g(t_a,\tau) = \pm\dot{G}_g(t_b,\tau) &\Longrightarrow \dot{G}_g^+(t_a,\tau) = \pm\dot{G}_g^-(t_b,\tau)\end{aligned}$$

which, after plugging in equation (3.3) and adding/subtracting the resulting equations, yields

$$\begin{bmatrix} \eta_g(t_a)+\dot{\eta}_g(t_a) & \xi_g(t_a)+x\dot{i}_g(t_a) \\ \eta_g(t_a)-\dot{\eta}_g(t_a) & \xi_g(t_a)-x\dot{i}_g(t_a) \end{bmatrix} \begin{bmatrix} a_g(\tau) \\ b_g(\tau) \end{bmatrix} = \pm \begin{bmatrix} \eta_g(t_b)+\dot{\eta}_g(t_b) & \xi_g(t_b)+x\dot{i}_g(t_b) \\ \eta_g(t_b)-\dot{\eta}_g(t_b) & \xi_g(t_b)-x\dot{i}_g(t_b) \end{bmatrix} \begin{bmatrix} c_g(\tau) \\ d_g(\tau) \end{bmatrix}.$$

This has a solution provided that, again, we have NO zero modes (the Wronskian $W_g \neq 0$) where the following determinant vanishes:

$$\det \bar{\Gamma_g} = \begin{vmatrix} \eta_g(t_a)+\dot{\eta}_g(t_a) & \xi_g(t_a)+x\dot{i}_g(t_a) \\ \eta_g(t_a)-\dot{\eta}_g(t_a) & \xi_g(t_a)-x\dot{i}_g(t_a) \end{vmatrix}.$$

If we introduce the function $\bar{\Delta_g}(t,\tau) = 2 \pm \partial_t \Delta_g(t,\tau) \pm \partial_t \Delta_g(\tau,t)$ then we see (after much tedious algebra) that our periodic/antiperiodic Green function $\bar{G_g}(t,\tau)$ is just (we implicitly introduce the bar notation to separate ourselves from the Dirichlet case)

$$\bar{G_g}(t,\tau) = G_g(t,\tau) \mp \frac{[\Delta_g(t,t_a) \pm \Delta_g(t_b,t)][\Delta_g(\tau,t_a) \pm \Delta_g(t_b,\tau)]}{\bar{\Delta_g}(t_a,t_b)\Delta_g(t_a,t_b)}. \tag{3.11}$$

4. Ratio of the Functional Determinant

Now we are ready to evaluate equation (2.6) using our explicit formulas for the Green functions. Here we follow the well-known method put forward by Gelfand and Yaglom [5].

4.1. Dirichlet Boundary Conditions.

We begin by considering the following equations

$$K_g(t)D_g(t) = 0; \quad D_g(t_a) = 0; \quad \dot{D}_g(t_a) = 1. \tag{4.1}$$

The function $D_g(t)$ will exhibit several desirable properties which will allow for an explicit solution of equation (2.6). Differentiating both sides with respect to the parameter g, we obtain the initial-value equations

$$K_g D'_g(t) = \Omega^2(t) D_g(t); \quad D'_g(t_a) = 0; \quad \dot{D}'_g(t_a) = 1 \tag{4.2}$$

where $D'_g \equiv \partial_g D_g(t)$.

Since equation (4.1) is just the homogeneous equation, we see that the solution is given in terms of $\eta_g(t)$ and $\xi_g(t)$ as

$$D_g(t) = \frac{\eta_g(t_a)\xi_g(t) - \xi_g(t_a)\eta_g(t)}{W_g} = \Delta_g(t_a,t) \tag{4.3}$$

which implies that (evaluating at $t = t_b$)

$$D_g(t_b) = \Delta_g(t_a,t_b) = \frac{\det \Lambda_g}{W_g}$$

where $\det \Lambda_g$ was given in equation (3.9).

The solution to equation (4.2) (using the Green function) is just

$$D'_g(t) = \int_{t_a}^{t} d\tau \Omega^2(\tau) D_g(\tau) G_g(t,\tau),$$

which, evaluating at t_b and using equations (3.10) and (4.3) yields

$$D'_g(t_b) = -\Delta_g(t_a,t_b) \int_{t_a}^{t_b} dt \Omega^2(t) G_g(t,t).$$

But this is just the expression

$$\frac{D_g'(t_b)}{D_g(t_b)} = -\int_{t_a}^{t_b} dt\Omega^2(t)G_g(t,t) = -\mathrm{Tr}\Omega^2(t)G_g(t,\tau).$$

Thus

$$\mathrm{Tr}\Omega^2(t)G_g(t,\tau) = -\partial_g \log D_g(t_b).$$

Plugging this result into equation (2.6) gives the compact result

$$\det \widetilde{K}^{-1}K_g = CD_g(t_b).$$

It only remains to determine C. But this is easily determined by letting $\widetilde{K} = -\partial_t^2 = K_0$ and considering the case where $g = 0$. Then the LHS is just unity, and in this simple case the explicit solution to equation (4.1) is just $t - t_a$, so we obtain our desired result

$$\det \widetilde{K}^{-1}K_1 = \frac{D_1(t_b)}{t_b - t_a}.$$

Thus we have determined the ratio of functional determinants in terms of a solution to the differential equation (4.1).

4.2. **Periodic/Antiperiodic Boundary Conditions.** For periodic and antiperiodic boundary conditions we can follow a similar route, although here we require more complication. To this end consider an equation with boundary conditions opposite to those in equation (4.1):

$$K_g(t)\bar{D}_g(t) = 0 \quad \bar{D}_g(t_a) = 1 \quad \dot{\bar{D}}_g(t_a) = 0. \tag{4.4}$$

Differentiating this with respect to g yields the same form found in equation (4.2), namely

$$K_g(\bar{t})D_g'(t) = \Omega^2(t)\bar{D}_g(t) \quad \bar{D}_g'(t_a) = 0 \quad \dot{\bar{D}}_g'(t_a) = 0. \tag{4.5}$$

It is easily checked that the solution to equation (4.4) is given by

$$\bar{D}_g(t) = \frac{\eta_g(t)\dot{\xi}_g(t_a) - \xi_g(t)\dot{\eta}_g(t_a)}{W_g} = -\dot{\Delta}_g(t_a,t).$$

The dot denotes differentiation with respect to the first variable. Hence the solution to equation (4.5) is given by

$$\bar{D}_g'(t) = -\int_{t_a}^{t} d\tau\Omega^2(\tau)\bar{D}_g(\tau)\bar{G}_g(t,\tau).$$

Evaluating at $t = t_b$ and considering equation (3.11) gives

$$\dot{D}_g'(t_b) + \bar{D}_g'(t_b) = \pm\bar{\Delta}_g(t_a,t_b)\int_{t_a}^{t_b} dt\Omega^2(t)\bar{G}_g(t,t)$$

which, after rearranging, yields the expression

$$\int_{t_a}^{t_b} dt\Omega^2(t)\bar{G}_g(t,t) = \mp\frac{\dot{D}_g'(t_b) + \bar{D}_g'(t_b)}{\bar{\Delta}_g(t_a,t_b)} = -\partial_g \log\left[2 \mp \dot{D}_g(t_b) \mp \bar{D}_g(t_b)\right].$$

Putting this into equation (2.6) implies

$$\det \widetilde{K}^{-1}K_g = C[2 \mp \dot{D}_g(t_b) \mp \bar{D}_g(t_b)], \tag{4.6}$$

so it remains to evaluate the constant C. We choose $\widetilde{K} = -\partial_t^2 - \omega^2$, and consider the case when $g = 1$. Letting $\Omega(t) = \omega$ as a special case (notice that the determinant above is constant, hence this is valid) implies that the LHS is unity. Solving the differential equations (4.1) and (4.4) yields

$$D_1(t) = \frac{1}{\omega}\sin(\omega(t-t_a)) \quad \bar{D_1}(t) = \cos(\omega(t-t_a))$$

which implies that

$$1 = \begin{array}{ll} 4C\sin^2(\omega(t-t_a)/2) & \text{periodic} \\ 4C\cos^2(\omega(t-t_a)/2) & \text{antiperiodic.} \end{array}$$

This determines the constant C completely and hence equation (4.6) is our desired result.

5. Conclusion

In this review we have followed the method of H. Kleinert and A. Chervyakov for calculating the ratio of functional determinants for second-order differential operators. In particular we have reduced these ratios to explicit formulas involving only homogeneous solutions of the corresponding second-order differential equations (with dual boundary conditions).

The formulas have been presented for both the cases of Dirichlet and periodic/antiperiodic boundary conditions. With a slight modification involving regularization this method can actually be extended to include operators with zero modes [2] (although we have only considered the case of operators without zero modes in this treatment).

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